2.050J/12.006J/18.353J Nonlinear Dynamics I: Chaos, Fall 2012

Problem Set 9

Due at 12:01 pm on Friday, November 16th, in the box provided. No late psets are accepted. If you collaborated with other students in the class, list their names on the title sheet. The work that you submit must be your own.

Main concepts: Matthieu's equation, Floquet theory

Reading: Lecture on Matthieu's equation is posted online

In class analyzed a vertically driven pendulum described by the Matthieu's equation. Physically this is a frictionless pendulum in oscillating gravitational field.

$$\ddot{\theta} + \omega_0^2(1 + h\cos(2\omega t))\theta = 0, \quad h \geq 0. \tag{1}$$

For the pset you have to study the stability of the periodic solution with friction, i.e. equation

$$\ddot{\theta} + \gamma\dot{\theta} + \omega_0^2(1 + h\cos(2\omega t))\theta = 0, \quad \gamma > 0. \tag{2}$$

1. Describe what the terms mean physically, consider $h = 0$ and $\gamma = 0$ cases.

2. (Numerical) Pick some values of the constants (e.g. $h = 0.2$, $\omega_0 = 1$), and look numerically for the typical examples of the behavior of the system (both as a time-evolution $\theta(t)$ and on the phase plane $(\theta, \dot{\theta})$). For the same initial conditions, plot in MATLAB both the time-evolution and the trajectory in phase plane for different values of γ (for examples, $\gamma = 0$, 0.1 and 0.3). Try ω close to ω_0 Describe which distinct physical regimes you have depending on the damping parameter, amplitude and frequency of the forcing.

   Note: this system has time-dependence in the coefficients.

3. (Analytical) Since after adding damping this is still a linear system with periodic coefficients, you can apply the Floquet theory. It states that the solution can be written in the form of an exponential function multiplying a periodic function, which we represent as a sum of cosines.

   $$\theta(t) = e^{\mu t}\sum_{n=1}^{\infty} a_n \cos(n\omega t + \Phi_n).$$

   Plug this expression in (2), collect the terms in front of a_1 and set them equal to zero. Explain the reason behind this.

   Drop the high frequency terms (as in lecture, when we dropped $\cos(3\omega t + \Phi)$). It can be shown that that term is negligible.

4. In the expression that you obtained, collect the terms in front of $\sin(\omega t)$ and $\cos(\omega t)$. Set both of these groups equal to zero. Explain the reason behind the step.

5. The two equations that you obtained can be analyzed as a system of two linear equations for $\cos(\phi)$ and $\sin(\phi)$. When does this system have a solution? Deduce the criterion for existence of a non-trivial (i.e. not identically zero) solution. This will be your fourth order equation for μ, the coefficients will depend on ω, ω_0, and h.

6. Find the boundary of instability, i.e. find where $\mu = 0$ as a function of h and ω/ω_0. To which side of that boundary will the solutions be *unstable*? Plot it on the $(h, \omega/\omega_0)$ plane for small h.

7. The criterion for instability in the undamped case derived in class was

$$h > 2\left|1 - \frac{\omega}{\omega_0}\right| \tag{3}$$

   Plot it in Matlab, denote the regions of stability, instability. For infinitesimally small h, what is the frequency at which the pendulum is unstable? How does this frequency relate to the frequency of unperturbed pendulum?

   Note: It is not the same. It's a different instability threshold than that of the undamped forced pendulum $\ddot{x} + \omega_0^2 x = \cos(2\omega t)$, for which the resonance happened when the forcing frequency 2ω was equal to the natural frequency ω_0. Because of the relation that you'll find, the instability is called *subharmonic*.

8. On the same Matlab plot, plot the boundary of instability for the damped case for two values of $\gamma/\omega_0 = 0.1$ and $\gamma/\omega_0 = 0.5$. Interpret the change of the instability domain physically.

9. Now that you analyzed the system analytically, check your numerical experiments in question 2 and explain the behavior that you saw given that now you have a new perspective.