14. Instability of Superposed Fluids

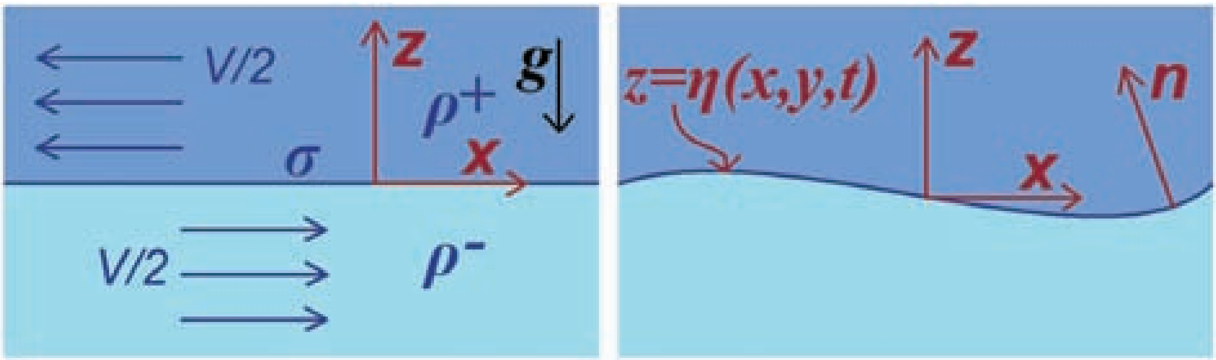

Figure 14.1: Wind over water: A layer of fluid of density ρ^+ moving with relative velocity V over a layer of fluid of density ρ^-.

Define interface: $h(x,y,z) = z - \eta(x,y) = 0$ so that $\boldsymbol{\nabla} h = (-\eta_x, -\eta_y, 1)$.
The unit normal is given by

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{\nabla} h}{|\boldsymbol{\nabla} h|} = \frac{(-\eta_x, -\eta_y, 1)}{\left(\eta_x^2 + \eta_y^2 + 1\right)^{1/2}} \tag{14.1}$$

Describe the fluid as inviscid and irrotational, as is generally appropriate at high $\mathcal{R}e$.
Basic state: $\eta = 0$, $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, $\phi = \mp\frac{1}{2}V_x$ for $z\pm$.
Perturbed state: $\phi = \mp\frac{1}{2}V_x + \phi_\pm$ in $z\pm$, where $\phi_\pm$ is the perturbation field.
Solve

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \nabla^2\phi_\pm = 0 \tag{14.2}$$

subject to BCs:

1. $\phi_\pm \to 0$ as $z \to \pm\infty$
2. **Kinematic BC:** $\frac{\partial\eta}{\partial t} = \boldsymbol{u}\cdot\boldsymbol{n}$,
   where
   $$\boldsymbol{u} = \boldsymbol{\nabla}\left(\mp\frac{1}{2}V_x + \phi_\pm\right) = \mp\frac{1}{2}V\hat{\boldsymbol{x}} + \frac{\partial\phi_\pm}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial\phi_\pm}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial\phi_\pm}{\partial z}\hat{\boldsymbol{z}} \tag{14.3}$$
   from which
   $$\frac{\partial\eta}{\partial t} = \left(\mp\frac{1}{2}V + \frac{\partial\phi_\pm}{\partial x}\right)(-\eta_x) + \frac{\partial\phi_\pm}{\partial y}(-\eta_y) + \frac{\partial\phi_\pm}{\partial z} \tag{14.4}$$
   Linearize: assume perturbation fields η, $\phi_\pm$ and their derivatives are small and therefore can neglect their products.
   Thus $\hat{\eta} \approx (-\eta_x, -\eta_y, 1)$ and $\frac{\partial\eta}{\partial t} = \pm\frac{1}{2}V\eta_x + \frac{\partial\phi_\pm}{\partial z} \Rightarrow$
   $$\frac{\partial\phi_\pm}{\partial z} = \frac{\partial\eta}{\partial t} \mp \frac{1}{2}V\frac{\partial\eta}{\partial x} \quad \textit{on } z = 0 \tag{14.5}$$
3. **Normal Stress Balance**: $p_- - p_+ = \sigma\boldsymbol{\nabla}\cdot\boldsymbol{n}$ on $z = \eta$.
   Linearize: $p_- - p_+ = -\sigma\left(\eta_{xx} + \eta_{yy}\right)$ on $z = 0$.

We now deduce $p_\pm$ from time-dependent Bernoulli:

$$\rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho u^2 + p + \rho g z = f(t) \tag{14.6}$$

where $u^2 = \frac{1}{4}V^2 \mp V\frac{\partial \phi_\pm}{\partial x} + H.O.T.$
Linearize:

$$\rho_\pm \frac{\partial \phi_\pm}{\partial t} + \frac{1}{2}\rho_\pm \left(\mp V \frac{\partial \phi_\pm}{\partial x}\right) + p_\pm + \rho_\pm g\eta = G(t) \tag{14.7}$$

so

$$p_- - p_+ = (\rho_+ - \rho_-)g\eta + (\rho_+ \frac{\partial \phi_\pm}{\partial t} - \rho_- \frac{\partial \phi_-}{\partial t}) + \frac{V}{2}(\rho_- \frac{\partial \phi_-}{\partial x} + \rho_+ \frac{\partial \phi_+}{\partial x}) = -\sigma(\eta_{xx} + \eta_{yy}) \tag{14.8}$$

is the linearized normal stress BC. Seek normal mode (wave) solutions of the form

$$\eta = \eta_0 e^{i\alpha x + i\beta y + \omega t} \tag{14.9}$$

$$\phi_\pm = \phi_{0\pm} e^{\mp kz} e^{i\alpha x + i\beta y + \omega t} \tag{14.10}$$

where $\nabla^2 \phi_\pm = 0$ requires $k^2 = \alpha^2 + \beta^2$.
Apply kinematic BC: $\frac{\partial \phi_\pm}{\partial z} = \frac{\partial \eta}{\partial t} \mp \frac{1}{2}V\frac{\partial \eta}{\partial x}$ at $z = 0 \Rightarrow$

$$\mp k\phi_{0\pm} = \omega\eta_0 \mp \frac{1}{2}i\alpha V \eta_0 \tag{14.11}$$

Normal stress BC:

$$k^2\sigma\eta_0 = -g(\rho_- - \rho_+)\eta_0 + \omega(\rho_+\phi_{0+} - \rho_-\phi_{0-}) + \frac{1}{2}i\alpha V\left(\rho_+\phi_{0+} + \rho_-\phi_{0-}\right) \tag{14.12}$$

Substitute for $\phi_{0\pm}$ from (14.11):

$$-k^3\sigma = \omega\left[\rho_+(\omega - \frac{1}{2}i\alpha V) + \rho_-(\omega + \frac{1}{2}i\alpha V)\right] + gk(\rho_- - \rho_+) + \frac{1}{2}i\alpha V\left[\rho_+(\omega - \frac{1}{2}i\alpha V) + \rho_-(\omega + \frac{1}{2}i\alpha V)\right]$$

so

$$\omega^2 + i\alpha V\left(\frac{\rho_- - \rho_+}{\rho_- + \rho_+}\right)\omega - \frac{1}{4}\alpha^2 V^2 + k^2 C_0^2 = 0 \tag{14.13}$$

where $C_0^2 \equiv \left(\frac{\rho_- - \rho_+}{\rho_- + \rho_+}\right)\frac{g}{k} + \frac{\sigma}{\rho_- + \rho_+}k$.
Dispersion relation: we now have the relation between ω and k

$$\omega = \frac{1}{2}i\left(\frac{\rho_+ - \rho_-}{\rho_- + \rho_+}\right)\boldsymbol{k}\cdot\boldsymbol{V} \pm \left[\frac{\rho_-\rho_+}{(\rho_- + \rho_+)^2}\left(\boldsymbol{k}\cdot\boldsymbol{V}\right)^2 - k^2C_0^2\right]^{1/2} \tag{14.14}$$

where $\boldsymbol{k} = (\alpha, \beta)$, $k^2 = \alpha^2 + \beta^2$.
The system is UNSTABLE if $\mathcal{R}e\,(\omega) > 0$, i.e. if

$$\frac{\rho_+\rho_-}{\rho_- + \rho_+}\left(\boldsymbol{k}\cdot\boldsymbol{V}\right)^2 > k^2C_0^2 \tag{14.15}$$

Squires Theorem:
Disturbances with wave vector $\boldsymbol{k} = (\alpha, \beta)$ parallel to $\boldsymbol{V}$ are most unstable. This is a general property of shear flows.

We proceed by considering two important special cases, Rayleigh-Taylor and Kelvin-Helmholtz instability.

14.1 Rayleigh-Taylor Instability

We consider an initially static system in which heavy fluid overlies light fluid: $\rho_+ > \rho_-$, $V = 0$. Via (14.15), the system is unstable if

$$C_0^2 = \frac{\rho_- - \rho_+}{\rho_+\rho_-}\frac{g}{k} + \frac{\sigma}{\rho_- + \rho_+}k < 0 \tag{14.16}$$

i.e. if $\rho_+ - \rho_- > \frac{\sigma k^2}{g} = \frac{4\pi^2\sigma}{g\lambda^2}$.
Thus, for instability, we require: $\lambda > 2\pi\lambda_c$ where $\lambda_c = \sqrt{\frac{\sigma}{\Delta\rho g}}$ is the capillary length.

Heuristic Argument:
Change in Surface Energy:
$\Delta E_S = \sigma \cdot \underbrace{\Delta l}_{arc\ length} = \sigma\left[\int_0^\lambda ds - \lambda\right] = \frac{1}{4}\sigma\epsilon^2 k^2\lambda.$
Change in gravitational potential energy:
$\Delta E_G = \int_0^\lambda -\frac{1}{2}\rho g\left(h^2 - h_0^2\right)dx = -\frac{1}{4}\rho g\epsilon^2\lambda.$
When is the total energy decreased?
When $\Delta E_{total} = \Delta E_S + \Delta E_G < 0$, i.e. when $\rho g > \sigma k^2$, so $\lambda > 2\pi l_c$.
The system is thus unstable to long λ.
Note:

1. The system is stabilized to small λ disturbances by σ
2. The system is always unstable for suff. large λ
3. In a finite container with width smaller than $2\pi\lambda_c$, the system may be stabilized by σ.
4. System may be stabilized by temperature gradients since Marangoni flow acts to resist surface deformation. E.g. a fluid layer on the ceiling may be stabilized by heating the ceiling.

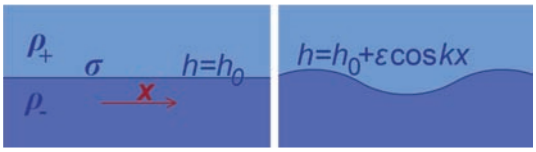

Figure 14.2: The base state and the perturbed state of the Rayleigh-Taylor system, heavy fluid over light.

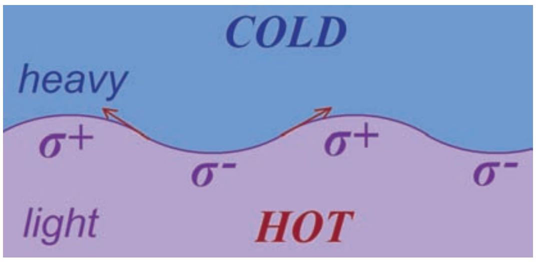

Figure 14.3: Rayleigh-Taylor instability may be stabilized by a vertical temperature gradient.

14.2 Kelvin-Helmholtz Instability

We consider shear-driven instability of a gravitationally stable base state. Specifically, $\rho_- \geq \rho_+$ so the system is gravitationally stable, but destabilized by the shear.
Take $\boldsymbol{k}$ parallel to $\boldsymbol{V}$, so $(\boldsymbol{V} \cdot \boldsymbol{k})^2 = k^2 V^2$ and the instability criterion becomes:

$$\rho_- \rho_+ V^2 > (\rho_- - \rho_+) \frac{g}{k} + \sigma k \tag{14.17}$$

Equivalently,

$$\rho_- \rho_+ V^2 > (\rho_- - \rho_+) g \frac{\lambda}{2\pi} + \sigma \frac{2\pi}{\lambda} \tag{14.18}$$

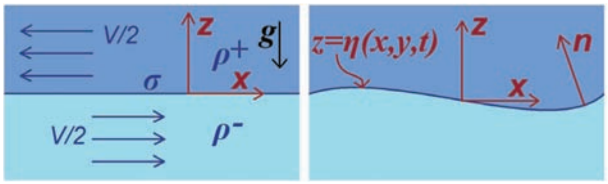

Figure 14.4: Kelvin-Helmholtz instability: a gravitationally stable base state is destabilized by shear.

`Note`:

1. System stabilized to short λ disturbances by surface tension and to long λ by gravity.
2. For any given λ (or k), one can find a critical V that destabilizes the system.

Marginal Stability Curve:

$$V(k) = \left(\frac{\rho_- - \rho_+}{\rho_- \rho_+} \frac{g}{k} + \frac{1}{\rho_- \rho_+} \sigma k \right)^{1/2} \tag{14.19}$$

$V(k)$ has a minimum where $\frac{dV}{dk} = 0$, i.e. $\frac{d}{dk} V^2 = 0$.
This implies $-\frac{\Delta \rho}{k^2} + \sigma = 0 \Rightarrow k_c = \sqrt{\frac{\Delta \rho g}{\sigma}} = \frac{1}{l_{cap}}$.
The corresponding $V_c = V(k_c) = \frac{2}{\rho_- \rho_+} \sqrt{\Delta \rho g \sigma}$ is the minimal speed necessary for waves.

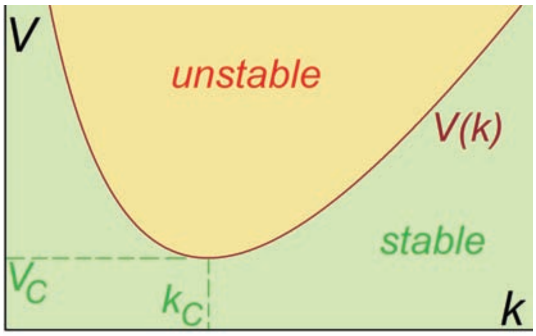

Figure 14.5: Fluid speed $V(k)$ required for the growth of a wave with wavenumber k.

E.g. Air blowing over water: (cgs)
$V_c^2 = \frac{2}{1.2 \cdot 10^{-3}} \sqrt{1 \cdot 10^3 \cdot 70} \Rightarrow V_c \sim 650$cm/s is the minimum wind speed required to generate waves.
These waves have wavenumber $k_c = \sqrt{\frac{1 \cdot 10^3}{70}} \approx 3.8\, cm^{-1}$, so $\lambda_c = 1.6$cm. They thus correspond to capillary waves.

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