19. Water waves

We consider waves that might arise from disturbing the surface of a pond.

We define the normal to the surface: $\boldsymbol{n} = \frac{(-\zeta_x, 1)}{(1+\zeta_x^2)^{1/2}}$
Curvature: $\boldsymbol{\nabla} \cdot \boldsymbol{n} = \frac{-\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}$
We assume the fluid motion is inviscid and irrotational: $\boldsymbol{u} = \boldsymbol{\nabla}\phi$. Must deduce solution for velocity potential ϕ satisfying $\nabla^2\phi = 0$.
B.C.s:
1. $\frac{\partial \phi}{\partial z} = 0$ on $z = -h$
2. Kinematic B.C.:
$\frac{D\zeta}{Dt} = u_z \Rightarrow \frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} = \frac{\partial \phi}{\partial z}$ on $z = \zeta$.
3. Dynamic B.C. (time-dependent Bernoulli applied at free surface):
$\rho\frac{\partial \phi}{\partial t} + \frac{1}{2}\rho\left|\boldsymbol{\nabla}\phi\right|^2 + \rho g \zeta + p_S = f(t)$, independent of x
where $p_s = p_0 + \sigma\boldsymbol{\nabla}\cdot\boldsymbol{n} = p_0 - \sigma\frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}$ is the surface pressure.

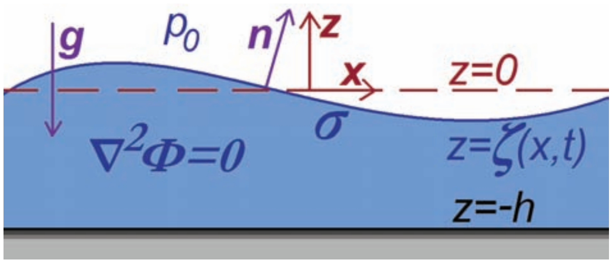

Figure 19.1: Waves on the surface of an inviscid irrotational fluid.

`Recall`: unsteady inviscid flows Navier-Stokes:

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho\left[\boldsymbol{\nabla}\left(\frac{1}{2}u^2\right) - \boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u})\right] = -\boldsymbol{\nabla}(p+\Psi) \tag{19.1}$$

For irrotational flows, $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, so that $\boldsymbol{u}\cdot\boldsymbol{\nabla}\left[\rho\frac{\partial \phi}{\partial t} + \frac{1}{2}\rho\left|\boldsymbol{\nabla}\phi\right|^2 + p + \Phi\right] = 0$.
Time-dependent Bernoulli: $\rho\frac{\partial \phi}{\partial t} + \frac{1}{2}\rho\left|\boldsymbol{\nabla}\phi\right|^2 + p + \Phi = F(t)$ only.

Now consider small amplitude waves and **linearize** the governing equations and BCs (assume ζ, ϕ are small, so we can neglect the nonlinear terms $\phi^2, \zeta^2, \phi\zeta$, etc.)
$\Rightarrow \nabla^2\phi = 0$ in $-h \le z \le 0$.
Must solve this equation subject to the **B.C.s**
1. $\frac{\partial \phi}{\partial z} = 0$ on $z = -h$
2. $\frac{\partial \zeta}{\partial t} = \frac{\partial \phi}{\partial z}$ on $z = 0$.
3. $\rho\frac{\partial \phi}{\partial t} + \rho g\zeta + p_0 - \sigma\zeta_{xx} = f(t)$ on $z = 0$.
Seek solutions: $\zeta(x,t) = \hat{\zeta}e^{ik(x-ct)}$, $\phi(x,z,t) = \hat{\phi}(z)e^{ik(x-ct)}$
i.e. travelling waves in x-direction with phase speed c and wavelength $\lambda = 2\pi/k$.
Substitute ϕ into $\nabla^2\phi = 0$ to obtain $\hat{\phi}_{zz} - k^2\hat{\phi} = 0$
Solutions: $\hat{\phi}(z) = e^{kz}, e^{-kz}$ or $\sinh(z), \cosh(z)$.
To satisfy B.C. 1: $\frac{\partial \hat{\phi}}{\partial z} = 0$ on $z = -h$ so choose $\hat{\phi}(z) = A\cosh k(z+h)$.
From B.C. 2:

$$ikc\hat{\zeta} = Ak\sinh kh \tag{19.2}$$

From B.C. 3: $\left(-ikc\rho A\cosh kh + \rho g\zeta + k^2\sigma\hat{\zeta}\right)e^{ik(x-ct)} = f(t)$, independent of x, i.e.

$$-ikc\rho A\cosh kh + \rho g\hat{\zeta} + k^2\sigma\hat{\zeta} = 0 \tag{19.3}$$

(19.2)$\Rightarrow A = \frac{ic\zeta}{\sinh kh} \Rightarrow$ into (19.3) $\Rightarrow c^2 = \left(\frac{g}{k} + \frac{\sigma k}{\rho}\right)\tanh kh$ defines the phase speed $c = \omega/k$.
Dispersion Relation:

$$\omega^2 = \left(gk + \frac{\sigma k^3}{\rho}\right)\tanh kh \tag{19.4}$$

`Note:` as $h \to \infty$. $\tanh kh \to 1$, and we obtain deep water dispersion relation deduced in our wind-over-water lecture.

Physical Interpretation

- relative importance of σ and g is prescribed by the Bond number $\mathbb{B}o = \frac{\rho g}{\sigma k^2} = \frac{\sigma(2\pi)^2}{\rho g \lambda^2} = (2\pi)^2 \frac{\ell_c^2}{\lambda^2}$ where $\ell_c = \sqrt{\sigma/\rho g}$ is the capillary length.
- for air-water, $\mathbb{B}o \sim 1$ for $\lambda \sim 2\pi\ell_c \sim 1.7$cm.
- $\mathbb{B}o \gg 1$, $\lambda \gg 2\pi\ell_c$: surface effects negligible $\Rightarrow$ gravity waves.
- $\mathbb{B}o \ll 1 : \lambda \ll 2\pi\ell_c$: influence of g is negligible $\Rightarrow$ capillary waves.

Special Cases: deep and shallow water. Can expand via Taylor series: For $kh \ll 1$, $\tanh kh = kh - \frac{1}{3}(kh)^3 + O\left((kh)^5\right)$, and for $kh \gg 1$, $\tanh kh \approx 1$.

A. Gravity waves $\mathbb{B}o \gg 1$: $c^2 = \frac{g}{k}\tanh kh$.
`Shallow water` $(kh \ll 1) \Rightarrow c = \sqrt{gh}$. All wavelengths travel at the same speed (i.e. non-dispersive), so one can only surf in shallow water.
`Deep water` $(kh \gg 1) \Rightarrow c = \sqrt{g/k}$, so longer waves travel faster, e.g. drop large stone into a pond.

B. Capillary Waves: $\mathbb{B}o \ll 1$, $c^2 = \frac{\sigma k}{\rho}\tanh kh$.

`Deep water` $kh \gg 1 \Rightarrow c = \sqrt{\sigma k \rho}$ so short waves travel fastest, e.g. raindrop in a puddle.

`Shallow water` $kh \ll 1 \Rightarrow c = \sqrt{\frac{\sigma h k^2}{\rho}}$.
An interesting note: in lab modeling of shallow water waves $(kh \ll 1)$ $c^2 \approx \left(\frac{g}{k} + \frac{\sigma k}{\rho}\right)\left(kh - \frac{1}{3}k^3h^3 + O\left((kh)^5\right)\right) = gh + \left(\frac{\sigma h}{\rho} - \frac{1}{3}gh^2\right)k^2 + O\left((kh)^4\right)gh$. In ripple tanks, choose $h = \left(\frac{3\sigma}{\rho g}\right)^{1/2}$ to get a good approximation to nondispersive waves. In water, $\left(\frac{3\sigma}{\rho g}\right)^{1/2} \sim \left(\frac{3\cdot 70}{10^3}\right)1/2 \sim 0.5$cm.

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Figure 19.2: Deep water capillary waves, whose speed increases as wavelength decreases.

From $c(k)$ can deduce $c_{min} = \left(\frac{4g\sigma}{\rho}\right)^{1/4}$ for $k_{min} = \left(\frac{\rho g}{\sigma}\right)^{1/2}$.
Group velocity: when $c = c(\lambda)$, a wave is called dispersive since its different Fourier components (corresponding to different k or λ) separate or disperse, e.g. deep water gravity waves: $c \sim \sqrt{\lambda}$. In a dispersive system, the energy of a wave component does not propagate at $c = \omega/k$ (phase speed), but at the **group velocity**:

$$c_g = \frac{d\omega}{dk} = \frac{d}{dk}(ck) \tag{19.5}$$

Deep gravity waves: $\omega = ck = \sqrt{gk}$. $c_g = \frac{\partial}{\partial k}\omega = \frac{\partial}{\partial k}\sqrt{gk} = \frac{1}{2}\sqrt{g/k} = \frac{c}{2}$.
Deep capillary wave: $c = \frac{\sigma/\rho}{k}^{1/2}$, $\omega = \sqrt{\sigma/\rho}k^{3/2} \Rightarrow c_g = \frac{\partial\omega}{\partial k} = \frac{3}{2}\sqrt{\sigma/\rho}k^{1/2} = \frac{3}{2}c$.

Flow past an obstacle.
If $U < c_{min}$, no steady waves are generated by the obstacle.
If $U > c_{min}$, there are two $k-$values, for which $c = U$:

1. the smaller k is a gravity wave with $c_g = c/2 < c \Rightarrow$ energy swept downstream.

2. the larger k is a capillary wave with $c_g = 3c/2 > c$, so the energy is swept upstream.

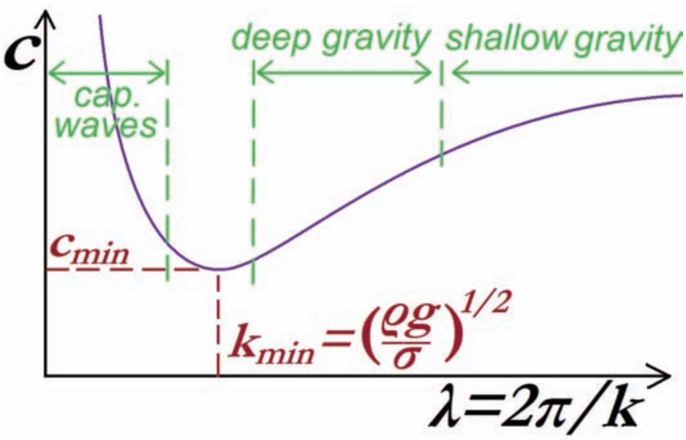

Figure 19.3: Phase speed c of surface waves as a function of their wavelength λ.

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