4. Young's Law with Applications

Young's Law: what is the equilibrium contact angle θ_e ? Horizontal force balance at contact line: $\gamma_{LV}\cos\theta_e = \gamma_{SV} - \gamma_{SL}$

$$\cos\theta_e = \frac{\gamma_{SV} - \gamma_{SL}}{\gamma_{LV}} = 1 + \frac{S}{\gamma_{LV}} \quad (Young\ 1805) \tag{4.1}$$

Note:

1. When $S \geq 0$, $\cos\theta_e \geq 1 \Rightarrow \theta_e$ undefined and spreading results.
2. Vertical force balance not satisfied at contact line $\Rightarrow$ dimpling of soft surfaces.
   E.g. bubbles in paint leave a circular rim.
3. The static contact angle need not take its equilibrium value $\Rightarrow$ there is a finite range of possible static contact angles.

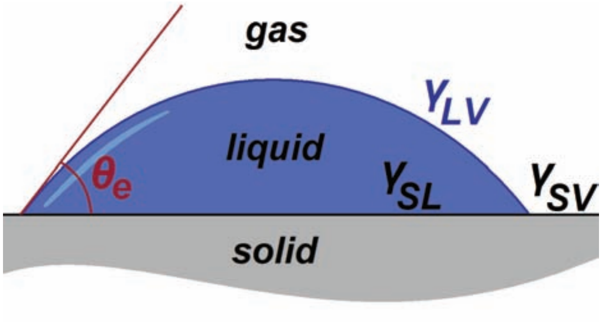

Figure 4.1: Three interfaces meet at the contact line.

Back to Puddles: Total energy:

$$E = \underbrace{(\gamma_{SL} - \gamma_{SV})A + \gamma_{LV}A}_{surface\ energy} + \underbrace{\frac{1}{2}\rho g h^2 A}_{grav.\ pot.\ energy} = -S\frac{V}{h} + \frac{1}{2}\rho g V h \tag{4.2}$$

Minimize energy w.r.t. h: $\frac{dE}{dh} = SV\frac{1}{h^2} + \frac{1}{2}\rho g V = 0$ when $-S/h^2 = \frac{1}{2}\rho g \Rightarrow$ $h_0 = \sqrt{\frac{-2S}{\rho g}} = 2\ell_c \sin\frac{\theta_e}{2}$ gives puddle depth, where $\ell_c = \sqrt{\sigma/\rho g}$.

Capillary Adhesion: Two wetted surfaces can stick together with great strength if $\theta_e < \pi/2$, e.g. Fig. 4.2.
Laplace Pressure:
$\Delta P = \sigma\left(\frac{1}{R} - \frac{\cos\theta_e}{H/2}\right) \approx -\frac{2\sigma\cos\theta_e}{H}$
i.e. low P inside film provided $\theta_e < \pi/2$.
If $H \ll R$, $F = \pi R^2 \frac{2\sigma\cos\theta_e}{H}$ is the attractive force between the plates.

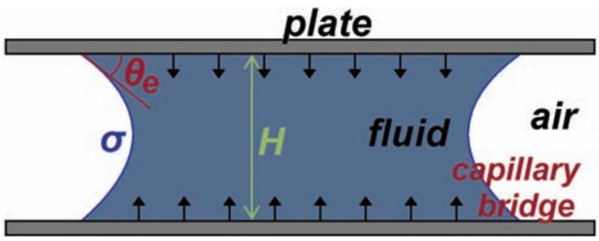

Figure 4.2: An oil drop forms a capillary bridge between two glass plates.

E.g. for H_2O, with $R = 1\ cm$, $H = 5\ \mu m$ and $\theta_e = 0$, one finds $\Delta P \sim 1/3$ atm and an adhesive force $F \sim 10N$, the weight of $1l$ of H_2O.

Note: Such capillary adhesion is used by beetles in nature.

4.1 Formal Development of Interfacial Flow Problems

Governing Equations: Navier-Stokes. An incompressible, homogeneous fluid of density ρ and viscosity $\mu = \rho\nu$ (μ is dynamic and ν kinematic viscosity) acted upon by an external force per unit volume $\mathbf{f}$ evolves according to

$$\nabla \cdot \boldsymbol{u} = 0 \quad (continuity) \tag{4.3}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \mathbf{f} + \mu\nabla^2\mathbf{u} \quad (Linear\ momentum\ conservation) \tag{4.4}$$

This is a system of 4 equations in 4 unknowns (u_1, u_2, u_3, p). These N-S equations must be solved subject to appropriate BCs.
Fluid-Solid BCs: "*No-slip*": $\mathbf{u} = \mathbf{U}_{solid}$.
`E.g.1` Falling sphere: $\mathbf{u} = \mathbf{U}$ on sphere surface, where $\mathbf{U}$ is the sphere velocity.
`E.g.2` Convection in a box: $\mathbf{u} = 0$ on the box surface.
But we are interested in flows dominated by interfacial effects. Here, in general, one must solve N-S equations in 2 domains, and match solutions together at the interface with appropriate BCs. Difficulty: These interfaces are free to move ⇒ Free boundary problems.

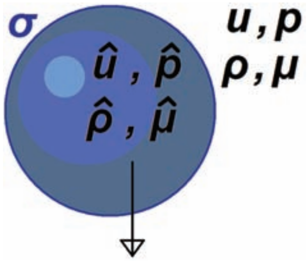

Figure 4.3: `E.g.3` Drop motion within a fluid.

Figure 4.4: `E.g.4` Water waves at an air-water interface.

Continuity of Velocity at an interface requires that $\mathbf{u} = \hat{\mathbf{u}}$.
And what about p ? We've seen $\Delta p \sim \sigma/R$ for a static bubble/drop, but to answer this question in general, we must develop stress conditions at a fluid-fluid interface.
`Recall`: **Stress Tensor**. The state of stress within an incompressible Newtonian fluid is described by the stress tensor: $\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{E}$ where $\mathbf{E} = \frac{1}{2}\left[(\nabla \boldsymbol{u}) + (\nabla \boldsymbol{u})^T\right]$ is the deviatoric stress tensor. The associated hydrodynamic force per unit volume within the fluid is $\nabla \cdot \boldsymbol{T}$.
One may thus write N-S eqns in the form: $\rho \frac{D\mathbf{u}}{Dt} = \nabla \cdot \boldsymbol{T} + \mathbf{f} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}$.
Now: T_{ij} = force / area acting in the $\mathbf{e}_j$ direction on a surface with a normal $\mathbf{e}_i$.

`Note`:

1. normal stresses (diagonals) T_{11}, T_{22}, T_{33} involve both p and u_i
2. tangential stresses (off-diagonals) T_{12}, T_{13}, etc., involve only velocity gradients, i.e. viscous stresses
3. T_{ij} is symmetric (Newtonian fluids)
4. $\mathbf{t}(\mathbf{n}) = \mathbf{n} \cdot \mathbf{T}$ = stress vector acting on a surface with normal $\mathbf{n}$

`E.g.` **Shear flow**. Stress in lower boundary is tangential. Force / area on lower boundary:
$T_{yx} = \mu \frac{\partial u_x}{\partial y}|_{y=0} = \mu k$ is the force/area that acts on y-surface in x-direction.

`Note`: the form of $\mathbf{T}$ in arbitrary curvilinear coordinates is given in the Appendix of Batchelor.

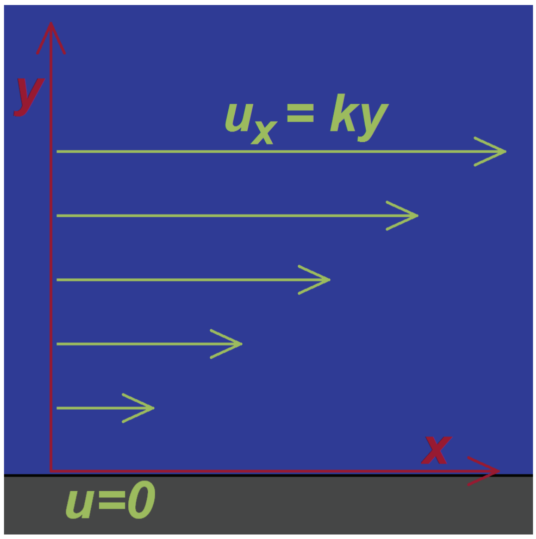

Figure 4.5: Shear flow above a rigid lower boundary.

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