5. Stress Boundary Conditions

Today:

1. Derive stress conditions at a fluid-fluid interface. Requires knowledge of $\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{E}$

2. Consider several examples of fluid statics

`Recall`: the curvature of a string under tension may support a normal force. *(see right)*

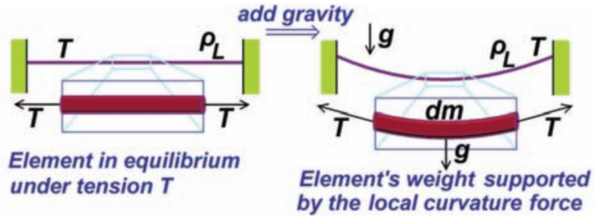

Figure 5.1: String under tension and the influence of gravity.

5.1 Stress conditions at a fluid-fluid interface

We proceed by deriving the normal and tangential stress boundary conditions appropriate at a fluid-fluid interface characterized by an interfacial tension σ.

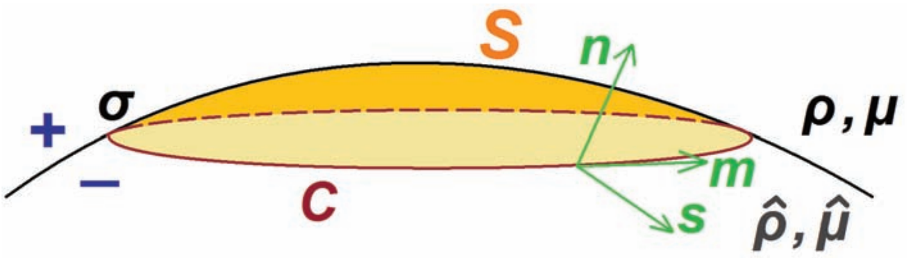

Figure 5.2: A surface S and bounding contour C on an interface between two fluids. Local unit vectors are $\boldsymbol{n}$, $\boldsymbol{m}$ and $\boldsymbol{s}$.

Consider an interfacial surface S bounded by a closed contour C. One may think of there being a force per unit length of magnitude σ in the s-direction at every point along C that acts to flatten the surface S. Perform a force balance on a volume element V enclosing the interfacial surface S defined by the contour C:

$$\int_V \rho \frac{D\mathbf{u}}{Dt} \mathrm{d}V = \int_V \mathbf{f}\, \mathrm{d}V + \int_{S^*} \left[t(\mathbf{n}) + \hat{t}(\hat{\mathbf{n}})\right] \mathrm{d}S + \int_C \sigma \mathbf{s}\, \mathrm{d}\ell \tag{5.1}$$

Here ℓ indicates arc-length and so $\mathrm{d}\ell$ a length increment along the curve C.
$\mathbf{t}(\mathbf{n}) = \boldsymbol{n} \cdot \boldsymbol{T}$ is the stress vector, the force/area exerted by the upper (+) fluid on the interface.
The stress tensor is defined in terms of the local fluid pressure and velocity field as $\mathbf{T} = -p\mathbf{I} + \mu\left[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T\right]$.
The stress exerted on the interface by the lower (-) fluid is $\hat{\mathbf{t}}(\hat{\mathbf{n}}) = \hat{\mathbf{n}} \cdot \hat{\mathbf{T}} = -\mathbf{n} \cdot \mathbf{T}$
where $\hat{\mathbf{T}} = -\hat{p}\mathbf{I} + \hat{\mu}\left[\boldsymbol{\nabla}\hat{\boldsymbol{u}} + (\boldsymbol{\nabla}\hat{\boldsymbol{u}})^T\right]$.

Physical interpretation of terms

$\int_V \rho \frac{D\mathbf{u}}{Dt}\, \mathrm{d}V$:	inertial force associated with acceleration of fluid in V
$\int \mathbf{f}\, \mathrm{d}V$:	body forces acting within V
$\int_S \mathbf{t}(\mathbf{n})\, \mathrm{d}S$:	hydrodynamic force exerted by upper fluid
$\int_S \hat{\mathbf{t}}(\hat{\mathbf{n}})\, \mathrm{d}S$:	hydrodynamic force exerted by lower fluid
$\int_C \sigma \mathbf{s}\, \mathrm{d}\ell$:	surface tension force exerted on perimeter.

Now if ϵ is the characteristic height of our volume V and R its characteristic radius, then the acceleration and body forces will scale as $R^2\epsilon$, while the surface forces will scale as R^2. Thus, in the limit of $\epsilon \to 0$, the latter must balance.

$$\int_S t(\mathbf{n}) + \hat{t}(\hat{\mathbf{n}})\ \mathrm{d}S + \int_C \sigma\mathbf{s}\ \mathrm{d}\ell = 0 \tag{5.2}$$

Now we have that

$$\mathbf{t}(\mathbf{n}) = \mathbf{n}\cdot\mathbf{T} \quad , \quad \hat{\mathbf{t}}(\hat{\mathbf{n}}) = \hat{\mathbf{n}}\cdot\hat{\mathbf{T}} = -\mathbf{n}\cdot\mathbf{T} \tag{5.3}$$

Moreover, the application of Stokes Theorem (see below) allows us to write

$$\int_C \sigma\mathbf{s}\ \mathrm{d}\ell = \int_S \nabla_S\sigma - \sigma\mathbf{n}\left(\nabla_S\cdot\mathbf{n}\right)\mathrm{d}S \tag{5.4}$$

where the tangential (surface) gradient operator, defined

$$\boldsymbol{\nabla}_S = [\mathbf{I} - \mathbf{nn}]\cdot\boldsymbol{\nabla} = \boldsymbol{\nabla} - \mathbf{n}\frac{\partial}{\partial\mathbf{n}} \tag{5.5}$$

appears because σ and $\mathbf{n}$ are only defined on the surface S. We proceed by dropping the subscript s on $\boldsymbol{\nabla}$, with this understanding. The surface force balance thus becomes

$$\int_S \left(\mathbf{n}\cdot\mathbf{T} - \mathbf{n}\cdot\hat{\mathbf{T}}\right)\mathrm{d}S = \int_S \sigma\mathbf{n}\left(\boldsymbol{\nabla}\cdot\boldsymbol{n}\right) - \boldsymbol{\nabla}\sigma\ \mathrm{d}S \tag{5.6}$$

Now since the surface S is arbitrary, the integrand must vanish identically. One thus obtains the interfacial stress balance equation, which is valid at every point on the interface:

Stress Balance Equation

$$\mathbf{n}\cdot\mathbf{T} - \mathbf{n}\cdot\hat{\mathbf{T}} = \sigma\mathbf{n}\left(\boldsymbol{\nabla}\cdot\boldsymbol{n}\right) - \boldsymbol{\nabla}\sigma \tag{5.7}$$

Interpretation of terms:

$\mathbf{n}\cdot\mathbf{T}$	stress (force/area) exerted by + on - (will generally have both $\perp$ and $\parallel$ components)
$\mathbf{n}\cdot\hat{\mathbf{T}}$	stress (force/area) exerted by - on + (will generally have both $\perp$ and $\parallel$ components)
$\sigma\mathbf{n}\left(\boldsymbol{\nabla}\cdot\boldsymbol{n}\right)$	normal curvature force per unit area associated with local curvature of interface, $\boldsymbol{\nabla}\cdot\boldsymbol{n}$
$\boldsymbol{\nabla}\sigma$	tangential stress associated with gradients in σ

Normal stress balance Taking $\mathbf{n}\cdot$(5.7) yields the normal stress balance

$$\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{n} - \mathbf{n}\cdot\hat{\mathbf{T}}\cdot\mathbf{n} = \sigma(\boldsymbol{\nabla}\cdot\mathbf{n}) \tag{5.8}$$

The jump in the normal stress across the interface is balanced by the curvature pressure.
`Note`: If $\boldsymbol{\nabla}\cdot\mathbf{n} \neq 0$, there must be a normal stress jump there, which generally involves both pressure and viscous terms.

Tangential stress balance Taking $\mathbf{d}\cdot$(5.7), where $\mathbf{d}$ is any linear combination of $\mathbf{s}$ and $\mathbf{m}$ (any tangent to S), yields the tangential stress balance at the interface:

$$\mathbf{n}\cdot\mathbf{T}\cdot\mathbf{d}-\mathbf{n}\cdot\hat{\mathbf{T}}\cdot\mathbf{d}=\nabla\sigma\cdot\mathbf{d} \tag{5.9}$$

Physical Interpretation

- LHS represents the jump in tangential components of the hydrodynamic stress at the interface
- RHS represents the tangential stress (Marangoni stress) associated with gradients in σ, as may result from gradients in temperature θ or chemical composition c at the interface since in general $\sigma=\sigma(\theta,c)$
- LHS contains only the non-diagonal terms of $\mathbf{T}$ - only the velocity gradients, not pressure; therefore any non-zero $\nabla\sigma$ at a fluid interface must *always* drive motion.

5.2 Appendix A : Useful identity

Recall Stokes Theorem:

$$\int_C \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{\ell}=\int_S \boldsymbol{n}\cdot(\nabla\wedge\boldsymbol{F})\,\mathrm{d}S \tag{5.10}$$

Along the contour C, $\mathrm{d}\boldsymbol{\ell}=\boldsymbol{m}\,\mathrm{d}\ell$, so that we have

$$\int_C \boldsymbol{F}\cdot\boldsymbol{m}\,\mathrm{d}\ell=\int_S \boldsymbol{n}\cdot(\nabla\wedge\boldsymbol{F})\,\mathrm{d}S \tag{5.11}$$

Now let $\boldsymbol{F}=\boldsymbol{f}\wedge\boldsymbol{b}$, where $\boldsymbol{b}$ is an arbitrary *constant* vector. We thus have

$$\int_C (\boldsymbol{f}\wedge\boldsymbol{b})\cdot\boldsymbol{m}\,\mathrm{d}\ell=\int_S \boldsymbol{n}\cdot(\nabla\wedge(\boldsymbol{f}\wedge\boldsymbol{b}))\,\mathrm{d}S \tag{5.12}$$

Now use standard vector identities to see $(\boldsymbol{f}\wedge\boldsymbol{b})\cdot\boldsymbol{m}=-\boldsymbol{b}\cdot(\boldsymbol{f}\wedge\boldsymbol{m})$ and

$$\nabla\wedge(\boldsymbol{f}\wedge\boldsymbol{b})=\boldsymbol{f}\,(\nabla\cdot\boldsymbol{b})-\boldsymbol{b}\,(\nabla\cdot\boldsymbol{f})+\boldsymbol{b}\cdot\nabla\boldsymbol{f}-\boldsymbol{f}\cdot\nabla\boldsymbol{b}=-\boldsymbol{b}\,(\nabla\cdot\boldsymbol{f})+\boldsymbol{b}\cdot\nabla\boldsymbol{f} \tag{5.13}$$

since $\boldsymbol{b}$ is a constant vector. We thus have

$$\boldsymbol{b}\cdot\int_C (\boldsymbol{f}\wedge\boldsymbol{m})\,\mathrm{d}\ell=\boldsymbol{b}\cdot\int_S [\boldsymbol{n}\,(\nabla\cdot\boldsymbol{f})-(\nabla\boldsymbol{f})\cdot\boldsymbol{n}]\,\mathrm{d}S \tag{5.14}$$

Since $\boldsymbol{b}$ is arbitrary, we thus have

$$\int_C (\boldsymbol{f}\wedge\boldsymbol{m})\,\mathrm{d}\ell=\int_S [\boldsymbol{n}\,(\nabla\cdot\boldsymbol{f})-(\nabla\boldsymbol{f})\cdot\boldsymbol{n}]\,\mathrm{d}S \tag{5.15}$$

We now choose $\boldsymbol{f}=\sigma\boldsymbol{n}$, and recall that $\boldsymbol{n}\wedge\boldsymbol{m}=-\boldsymbol{s}$. One thus obtains $-\int_C \sigma\boldsymbol{s}\mathrm{d}\ell=\int_S [\boldsymbol{n}\nabla\cdot(\sigma\boldsymbol{n})-\nabla(\sigma\boldsymbol{n})\cdot\boldsymbol{n}]\,\mathrm{d}S=\int_S [\boldsymbol{n}\nabla\sigma\cdot\boldsymbol{n}+\sigma\boldsymbol{n}\,(\nabla\cdot\boldsymbol{n})-\nabla\sigma-\sigma\,(\nabla\boldsymbol{n})\cdot\boldsymbol{n}]\,\mathrm{d}S$.

We note that $\nabla\sigma\cdot\boldsymbol{n}=0$ since $\nabla\sigma$ must be tangent to the surface S and $(\nabla\boldsymbol{n})\cdot\boldsymbol{n}=\frac{1}{2}\nabla(\boldsymbol{n}\cdot\boldsymbol{n})=\frac{1}{2}\nabla(1)=0$, and so obtain the desired result:

$$\int_C \sigma\boldsymbol{s}\,\mathrm{d}\ell=\int_S [\nabla\sigma-\sigma\boldsymbol{n}\,(\nabla\cdot\boldsymbol{n})]\ \mathrm{d}S \tag{5.16}$$

5.3 Fluid Statics

We begin by considering static fluid configurations, for which the stress tensor reduces to the form $\mathbf{T} = -p\mathbf{I}$, so that $\mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n} = -p$, and the normal stress balance equation (5.8) assumes the simple form:

$$\hat{p} - p = \sigma \boldsymbol{\nabla} \cdot \boldsymbol{n} \tag{5.17}$$

The pressure jump across a static interface is balanced by the curvature force at the interface. Now since $\boldsymbol{n} \cdot \boldsymbol{T} \cdot \boldsymbol{d} = 0$ for a static system, the tangential stress balance indicates that $\boldsymbol{\nabla}\sigma = 0$. This leads to the following important conclusion: *There cannot be a static system in the presence of surface tension gradients.* While pressure jumps can sustain normal stress jumps across a fluid interface, they do not contribute to the tangential stress jump. Consequently, tangential surface (Marangoni) stresses can only be balanced by viscous stresses associated with fluid motion. We proceed by applying equation (5.17) to describe a number of static situations.

1. Stationary Bubble : We consider a spherical air bubble of radius R submerged in a static fluid. What is the pressure drop across the bubble surface?
The divergence in spherical coordinates of $\boldsymbol{F} = (F_r, F_\theta, F_\phi)$ is given by
$\boldsymbol{\nabla} \cdot \boldsymbol{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta F_\theta\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial\phi}F_\phi$.
Hence $\boldsymbol{\nabla} \cdot \boldsymbol{n}|_S = \frac{1}{r^2}\frac{\partial}{\partial r} r^2|_{r=R} = \frac{2}{R}$ so the normal stress jump (5.17) indicates that

$$\Delta P = \hat{p} - p = \frac{2\sigma}{R} \tag{5.18}$$

The pressure within the bubble is higher than that outside by an amount proportional to the surface tension, and inversely proportional to the bubble size. As noted in Lec. 2, it is thus that small bubbles are louder than large ones when they burst at a free surface: champagne is louder than beer. We note that soap bubbles in air have two surfaces that define the inner and outer surfaces of the soap film; consequently, the pressure differential is twice that across a single interface.

2. The static meniscus $(\theta_e < \pi/2)$

Consider a situation where the pressure within a static fluid varies owing to the presence of a gravitational field, $p = p_0 + \rho g z$, where p_0 is the constant ambient pressure, and $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$ is the grav. acceleration. The normal stress balance thus requires that the interface satisfy the *Young-Laplace Equation*:

$$\rho g z = \sigma \boldsymbol{\nabla} \cdot \boldsymbol{n} \tag{5.19}$$

The vertical gradient in fluid pressure must be balanced by the curvature pressure; as the gradient is constant, the curvature must likewise increase linearly with z. Such a situation arises in the static meniscus (*below*).

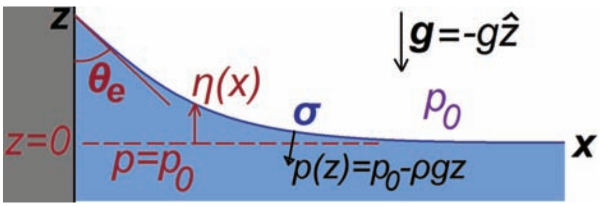

Figure 5.3: Static meniscus near a wall.

The shape of the meniscus is prescribed by two factors: the contact angle between the air-water interface and the wall, and the balance between hydrostatic pressure and curvature pressure. We treat the contact angle θ_e as given; noting that it depends in general on the surface energy. The normal force balance is expressed by the Young-Laplace equation, where now $\rho = \rho_w - \rho_{air} \approx \rho_w$ is the density difference between water and air. We define the free surface by $z = \eta(x)$; equivalently, we define a functional $f(x,z) = z - \eta(x)$ that vanishes on the surface. The normal to the surface $z = \eta(x)$ is thus

$$\boldsymbol{n} = \frac{\boldsymbol{\nabla} f}{|\boldsymbol{\nabla} f|} = \frac{\hat{\boldsymbol{z}} - \eta'(x)\hat{\boldsymbol{x}}}{\left[1 + \eta'(x)^2\right]^{1/2}} \tag{5.20}$$

As deduced in Appendix B, the curvature of the free surface $\boldsymbol{\nabla}\cdot\hat{\boldsymbol{n}}$, may be expressed as

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{n}} = \frac{-\eta_{xx}}{(1+\eta_x^2)^{3/2}} \approx -\eta_{xx} \tag{5.21}$$

Assuming that the slope of the meniscus remains sufficiently small, $\eta_x^2 \ll 1$, allows one to linearize equation (5.21), so that (5.19) assumes the form

$$\rho g \eta = \sigma \eta_{xx} \tag{5.22}$$

Applying the boundary condition $\eta(\infty) = 0$ and the contact condition $\eta_x(0) = -\cot\theta$, and solving (5.22) thus yields

$$\eta(x) = \ell_c \cot\theta_e e^{-x/\ell_c} \tag{5.23}$$

where $\ell_c = \sqrt{\sigma/\rho g}$ is the capillary length. The meniscus formed by an object floating in water is exponential, decaying over a length scale ℓ_c. Note that this behaviour may be rationalized as follows: the system arranges itself so that its total energy (grav. potential + surface) is minimized.

3. Floating Bodies

Floating bodies must be supported by some combination of buoyancy and curvature forces. Specifically, since the fluid pressure beneath the interface is related to the atmospheric pressure p_0 above the interface by

$$p = p_0 + \rho g z + \sigma \boldsymbol{\nabla}\cdot\boldsymbol{n}\ , \tag{5.24}$$

one may express the vertical force balance as

$$Mg = \boldsymbol{z}\cdot\int_C -p\boldsymbol{n}\,\mathrm{d}\ell = \underbrace{F_b}_{buoyancy} + \underbrace{F_c}_{curvature}\ . \tag{5.25}$$

The buoyancy force

$$F_b = \boldsymbol{z}\cdot\int_C \rho g z \boldsymbol{n}\ \mathrm{d}\ell = \rho g V_b \tag{5.26}$$

is thus simply the weight of the fluid displaced above the object and inside the line of tangency (*see figure below*). We note that it may be deduced by integrating the curvature pressure over the contact area C using the first of the Frenet-Serret equations (see Appendix C).

$$F_c = \boldsymbol{z}\cdot\int_C \sigma\left(\boldsymbol{\nabla}\cdot\boldsymbol{n}\right)\boldsymbol{n}\ \mathrm{d}\ell = \sigma\boldsymbol{z}\cdot\int_C \frac{d\boldsymbol{t}}{\mathrm{d}\ell}\ \mathrm{d}\ell = \sigma\boldsymbol{z}\cdot(\boldsymbol{t_1}-\boldsymbol{t_2}) = 2\sigma\sin\theta \tag{5.27}$$

At the interface, the buoyancy and curvature forces must balance precisely, so the Young-Laplace relation is satisfied:

$$0 = \rho g z + \sigma\boldsymbol{\nabla}\cdot\boldsymbol{n} \tag{5.28}$$

Integrating this equation over the meniscus and taking the vertical component yields the vertical force balance:

$$F_b^m + F_c^m = 0 \tag{5.29}$$

where

$$F_b^m = \boldsymbol{z}\cdot\int_{C_m} \rho g z \boldsymbol{n}\ \mathrm{d}\ell = \rho g V_m \tag{5.30}$$

$$F_c^m = \boldsymbol{z}\cdot\int_{C_m} \sigma\left(\boldsymbol{\nabla}\cdot\boldsymbol{n}\right)\boldsymbol{n}\ \mathrm{d}\ell = \sigma\boldsymbol{z}\cdot\int_{C_m} \frac{d\boldsymbol{t}}{\mathrm{d}\ell}\mathrm{d}\ell = \sigma\boldsymbol{z}\cdot(\boldsymbol{t_1}-\boldsymbol{t_2}) = -2\sigma\sin\theta \tag{5.31}$$

where we have again used the Frenet-Serret equations to evaluate the curvature force.

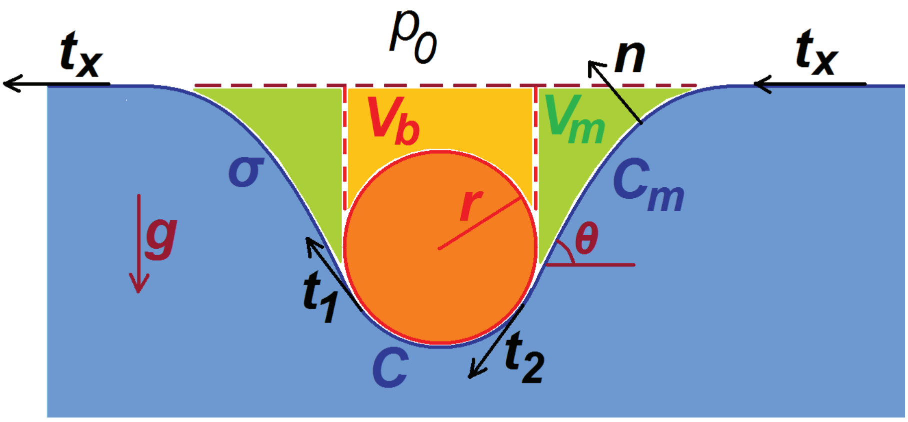

Figure 5.4: A floating non-wetting body is supported by a combination of buoyancy and curvature forces, whose relative magnitude is prescribed by the ratio of displaced fluid volumes V_b and V_m.

Equations (5.27-5.31) thus indicate that the curvature force acting on the floating body is expressible in terms of the fluid volume displaced *outside* the line of tangency:

$$F_c = \rho g V_m \tag{5.32}$$

The relative magnitude of the buoyancy and curvature forces supporting a floating, non-wetting body is thus prescribed by the relative magnitudes of the volumes of the fluid displaced inside and outside the line of tangency:

$$\frac{F_b}{F_c} = \frac{V_b}{V_m} \tag{5.33}$$

For 2D bodies, we note that since the meniscus will have a length comparable to the capillary length, $\ell_c = (\sigma/(\rho g))^{1/2}$, the relative magnitudes of the buoyancy and curvature forces,

$$\frac{F_b}{F_c} \approx \frac{r}{\ell_c} \; , \tag{5.34}$$

is prescribed by the relative magnitudes of the body size and capillary length. Very small floating objects ($r \ll \ell_c$) are supported principally by curvature rather than buoyancy forces. This result has been extended to three-dimensional floating objects by *Keller 1998, Phys. Fluids, 10, 3009-3010.*

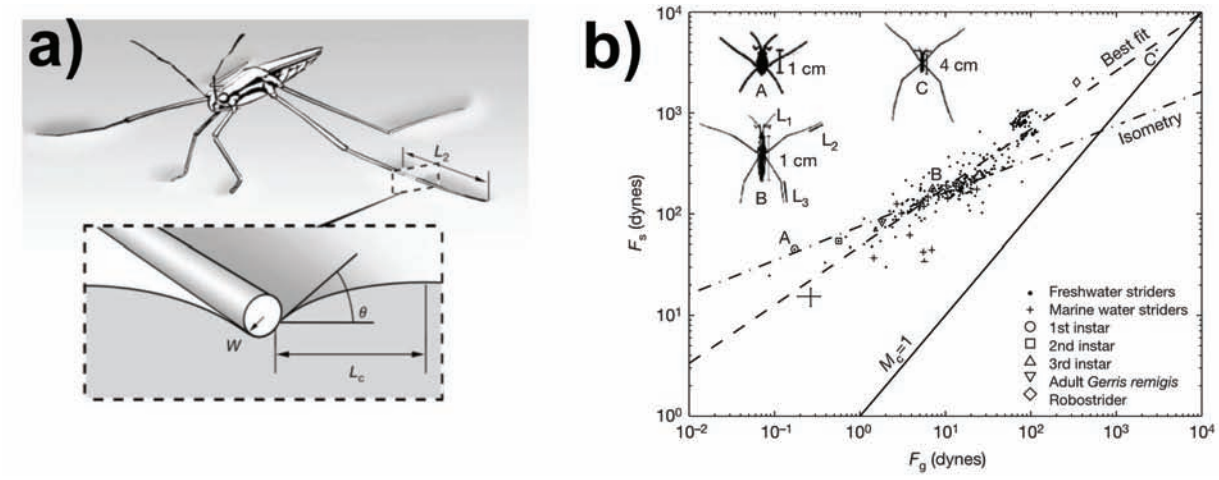

Figure 5.5: **a)** Water strider legs are covered with hair, rendering them effectively non-wetting. The tarsal segment of its legs rest on the free surface. The free surface makes an angle θ with the horizontal, resulting in an upward curvature force per unit length $2\sigma \sin\theta$ that bears the insect's weight. **b)** The relation between the maximum curvature force $F_s = 2\sigma P$ and body weight $F_g = Mg$ for 342 species of water striders. $P = 2(L_1 + L_2 + L_3)$ is the combined length of the tarsal segments. From *Hu, Chan & Bush; Nature 424, 2003.*

4. Water-walking Insects

Small objects such as paper clips, pins or insects may reside at rest on a free surface provided the curvature force induced by their deflection of the free surface is sufficient to bear their weight (Fig. 5.5a). For example, for a body of contact length L and total mass M, static equilibrium on the free surface requires that:

$$\frac{Mg}{2\sigma L \sin\theta} < 1 , \tag{5.35}$$

where θ is the angle of tangency of the floating body.
This simple criterion is an important geometric constraint on water-walking insects. Fig. 5.5b indicates the dependence of contact length on body weight for over 300 species of water-striders, the most common water walking insect. Note that the solid line corresponds to the requirement (5.35) for static equilibrium. Smaller insects maintain a considerable margin of safety, while the larger striders live close to the edge. The maximum size of water-walking insects is limited by the constraint (5.35).

If body proportions were independent of size L, one would expect the body weight to scale as L^3 and the curvature force as L. Isometry would thus suggest a dependence of the form $F_c \sim F_g^{1/3}$, represented as the dashed line. The fact that the best fit line has a slope considerably larger than 1/3 indicates a variance from isometry: the legs of large water striders are proportionally longer.

5.4 Appendix B : Computing curvatures

We see the appearance of the divergence of the surface normal, $\nabla \cdot \boldsymbol{n}$, in the normal stress balance. We proceed by briefly reviewing how to formulate this curvature term in two common geometries.

In cartesian coordinates (x, y, z), we consider a surface defined by $z = h(x, y)$. We define a functional $f(x, y, z) = z - h(x, y)$ that necessarily vanishes on the surface. The normal to the surface is defined by

$$\boldsymbol{n} = \frac{\nabla f}{|\nabla f|} = \frac{\hat{\boldsymbol{z}} - h_x \hat{\boldsymbol{x}} - h_y \hat{\boldsymbol{y}}}{\left(1 + h_x^2 + h_y^2\right)^{1/2}} \tag{5.36}$$

and the local curvature may thus be computed:

$$\nabla \cdot \boldsymbol{n} = \frac{-\left(h_{xx} + h_{yy}\right) - \left(h_{xx} h_y^2 + h_{yy} h_x^2\right) + 2 h_x h_y h_{xy}}{\left(1 + h_x^2 + h_y^2\right)^{3/2}} \tag{5.37}$$

In the simple case of a two-dimensional interface, $z = h(x)$, these results assume the simple forms:

$$\boldsymbol{n} = \frac{\hat{\boldsymbol{z}} - h_x \hat{\boldsymbol{x}}}{\left(1 + h_x^2\right)^{1/2}} \quad , \quad \nabla \cdot \boldsymbol{n} = \frac{-h_{xx}}{\left(1 + h_x^2\right)^{3/2}} \tag{5.38}$$

Note that $\boldsymbol{n}$ is dimensionless, while $\nabla \cdot \boldsymbol{n}$ has the units of $1/L$.

In 3D polar coordinates (r, θ, z), we consider a surface defined by $z = h(r, \theta)$. We define a functional $g(r, \theta, z) = z - h(r, \theta)$ that vanishes on the surface, and compute the normal:

$$\boldsymbol{n} = \frac{\nabla g}{|\nabla g|} = \frac{\hat{\boldsymbol{z}} - h_r \hat{\boldsymbol{r}} - \frac{1}{r} h_\theta \hat{\boldsymbol{\theta}}}{\left(1 + h_r^2 + \frac{1}{r^2} h_\theta^2\right)^{1/2}} \quad , \tag{5.39}$$

from which the local curvature is computed:

$$\nabla \cdot \boldsymbol{n} = \frac{-h_{\theta\theta} - h_r^2 h_{\theta\theta} + h_r h_{theta} - r h_r - \frac{2}{r} h_r h_\theta^2 - r^2 h_{rr} - h_{rr} h_\theta^2 + h_r h_\theta h_{r\theta}}{r^2 \left(1 + h_r^2 + \frac{1}{r^2} h_\theta^2\right)^{1/2}} \tag{5.40}$$

In the case of an axisymmetric interface, $z = h(r)$, these reduce to:

$$\boldsymbol{n} = \frac{\hat{\boldsymbol{z}} - h_r \hat{\boldsymbol{r}}}{\left(1 + h_r^2\right)^{1/2}} \quad , \quad \nabla \cdot \boldsymbol{n} = \frac{-r h_r - r^2 h_{rr}}{r^2 \left(1 + h_r^2\right)^{3/2}} \tag{5.41}$$

5.5 Appendix C : Frenet-Serret Equations

Differential geometry yields relations that are often useful in computing curvature forces on 2D interfaces.

$$\left(\nabla \cdot \boldsymbol{n}\right) \boldsymbol{n} = \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}\ell} \tag{5.42}$$

$$-\left(\nabla \cdot \boldsymbol{n}\right) \boldsymbol{t} = \frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}\ell} \tag{5.43}$$

Note that the LHS of (5.42) is proportional to the curvature pressure acting on an interface. Therefore the net force acting on surface S as a result of curvature / Laplace pressures:
$\mathbf{F} = \int_C \sigma \left(\nabla \cdot \boldsymbol{n}\right) \boldsymbol{n} \, \mathrm{d}\ell = \sigma \int_C \frac{\mathrm{d}\mathbf{t}}{\mathrm{d}\ell} \, \mathrm{d}\ell = \sigma \left(\boldsymbol{t_2} - \boldsymbol{t_1}\right)$
and so the net force on an interface resulting from curvature pressure can be deduced in terms of the geometry of the end points.

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