6. More on Fluid statics

Last time, we saw that the balance of curvature and hydrostatic pressures requires
$-\rho g\eta = \sigma \boldsymbol{\nabla}\cdot\boldsymbol{n} = \sigma\frac{-\eta_{xx}}{(1+\eta_x^2)^{3/2}}$.
We linearized, assuming $\eta_x \ll 1$, to find $\eta(x)$. `Note`: we can integrate directly

$$\rho g\eta\eta_x = \sigma\frac{\eta_x\eta_{xx}}{(1+\eta_x^2)^{3/2}}\rho g \Rightarrow \frac{d}{dx}\left(\frac{\eta^2}{2}\right) = \sigma\frac{d}{dx}\frac{1}{(1+\eta_x^2)^{1/2}} \Rightarrow$$
$$\frac{1}{2\sigma}\rho g\eta^2 = \int_x^\infty \frac{d}{dx}\frac{1}{(1+\eta_x^2)^{1/2}}dx = 1-\frac{1}{(1+\eta_x^2)^{1/2}} = 1-\sin\theta$$

$$\sigma\sin\theta + \frac{1}{2}\rho g\eta^2 = \sigma \tag{6.1}$$

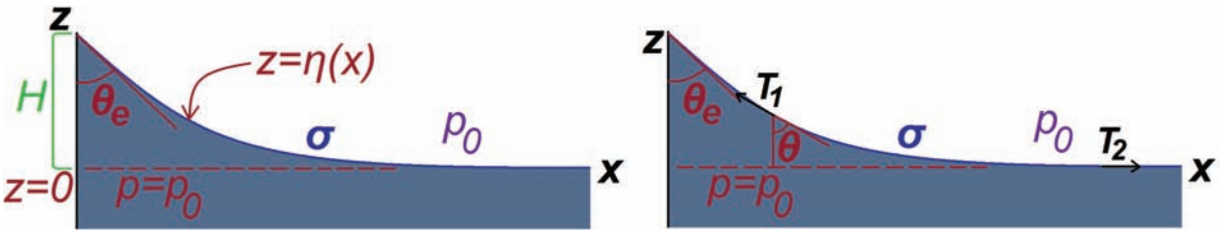

Figure 6.1: Calculating the shape and maximal rise height of a static meniscus.

Maximal rise height: At $z=h$ we have $\theta=\theta_e$, so from (6.1) $\frac{1}{2}\rho g h^2 = \sigma(1-\sin\theta_e)$, from which

$$h = \sqrt{2}\ell_c(1-\sin\theta_e)^{1/2} \quad \texttt{where} \quad \ell_c = \sqrt{\sigma/\rho g} \tag{6.2}$$

Alternative perspective: Consider force balance on the meniscus.
Horizontal force balance:

$$\underbrace{\sigma\sin\theta}_{horiz.\ projection\ of\ T_1} + \underbrace{\frac{1}{2}\rho g z^2}_{hydrostatic\ suction} = \underbrace{\sigma}_{T_2} \tag{6.3}$$

Vertical force balance:

$$\underbrace{\sigma\cos\theta}_{vert.\ proj.\ of\ T_1} = \underbrace{\int_x^\infty \rho g z dx}_{weight\ of\ fluid} \tag{6.4}$$

At $x=0$, where $\theta=\theta_e$, gives $\sigma\cos\theta_e =$ weight of fluid displaced above $z=0$.

`Note`: $\sigma\cos\theta_e =$ weight of displaced fluid is $+/-$ according to whether θ_e is smaller or larger than $\frac{\pi}{2}$.
Floating Bodies Without considering interfacial effects, one anticipates that heavy things sink and light things float. This doesn't hold for objects small relative to the capillary length.
`Recall`: Archimedean force on a submerged body $F_A = \int_S p\boldsymbol{n}dS = \rho\boldsymbol{g}V_B$.
In general, the hydrodynamic force acting on a body in a fluid
$\boldsymbol{F_h} = \int_S \boldsymbol{T}\cdot\boldsymbol{n}dS$, where $\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{E} = -p\mathbf{I}$ for static fluid.
Here $\mathbf{F}_h = -\int_S p\mathbf{n}dS = -\int_S \rho g z\mathbf{n}dS = -\rho g\int_V \boldsymbol{\nabla}z\ \mathrm{d}V$ by divergence theorem. This is equal to $-\rho g\int_V \mathrm{d}V\hat{\boldsymbol{z}} = -\rho gV\hat{\boldsymbol{z}} =$ weight of displaced fluid. The archimedean force can thus support weight of a body $Mg = \rho_B gV$ if $\rho_F > \rho_B$ (fluid density larger than body density); otherwise, it sinks.

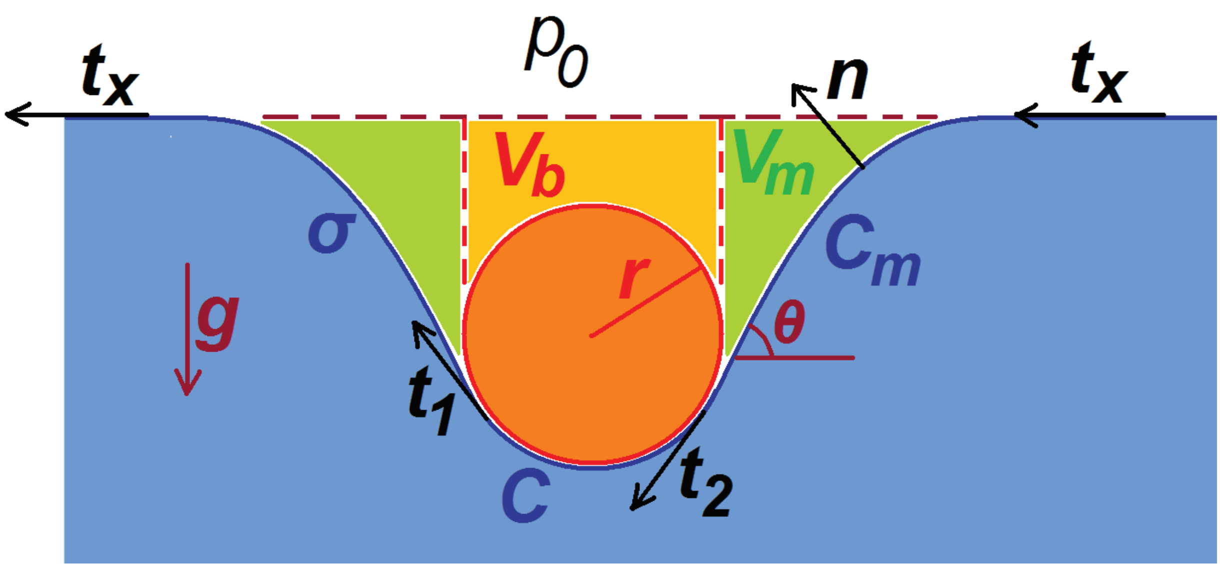

Figure 6.2: A heavy body may be supported on a fluid surface by a combination of buoyancy and surface tension.

6.1 Capillary forces on floating bodies

- arise owing to interaction of the menisci of floating bodies
- attractive or repulsive depending on whether the menisci are of the same or opposite sense
- explains the formation of bubble rafts on champagne
- explains the mutual attraction of Cheerios and their attraction to the walls
- utilized in technology for self-assembly on the microscale

Capillary attraction Want to calculate the attractive force between two floating bodies separated by a distance R. Total energy of the system is given by

$$E_{tot} = \sigma \oint \mathrm{d}A(R) + \int_{-\infty}^{\infty} \mathrm{d}x \int_0^{h(x)} \rho g z \mathrm{d}z \tag{6.5}$$

where the first term in (6.5) corresponds to the total surface energy when the two bodies are a distance R apart, and the second term is the total gravitational potential energy of the fluid. Differentiating (6.5) yields the force acting on each of the bodies:

$$F(R) = -\frac{\mathrm{d}E_{tot}(R)}{\mathrm{d}R} \tag{6.6}$$

Such capillary forces are exploited by certain water walking insects to climb menisci. By deforming the free surface, they generate a lateral force that drives them up menisci (*Hu & Bush 2005*).

357 Interfacial Phenomena
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