8. Capillary Rise

Capillary rise is one of the most well-known and vivid illustrations of capillarity. It is exploited in a number of biological processes, including drinking strategies of insects, birds and bats and plays an important role in a number of geophysical settings, including flow in porous media such as soil or sand.

`Historical Notes`:

- *Leonardo da Vinci* (1452 - 1519) recorded the effect in his notes and proposed that mountain streams may result from capillary rise through a fine network of cracks
- *Jacques Rohault* (1620-1675): erroneously suggested that capillary rise is due to suppresion of air circulation in narrow tube and creation of a vacuum
- *Geovanni Borelli* (1608-1675): demonstrated experimentally that $h \sim 1/r$
- *Geminiano Montanari* (1633-87): attributed circulation in plants to capillary rise
- *Francis Hauksbee* (1700s): conducted an extensive series of capillary rise experiments reported by Newton in his Opticks but was left unattributed
- *James Jurin* (1684-1750): an English physiologist who independently confirmed $h \sim 1/r$; hence "Jurin's Law".

Consider capillary rise in a cylindrical tube of inner radius a (Fig. 8.2)

`Recall`:
Spreading parameter: $S = \gamma_{SV} - (\gamma_{SL} + \gamma_{LV})$.

We now define Imbibition / Impregnation parameter:
$I = \gamma_{SV} - \gamma_{SL} = \gamma_{LV} \cos\theta$
via force balance at contact line.
`Note`: in capillary rise, I is the relevant parameter, since motion of the contact line doesn't change the energy of the liquid-vapour interface.

Imbibition Condition: $I > 0$.

`Note`: since $I = S + \gamma_{LV}$, the imbibition condition $I > 0$ is always more easily met than the spreading condition, $S > 0$
⇒ most liquids soak sponges and other porous media, while complete spreading is far less common.

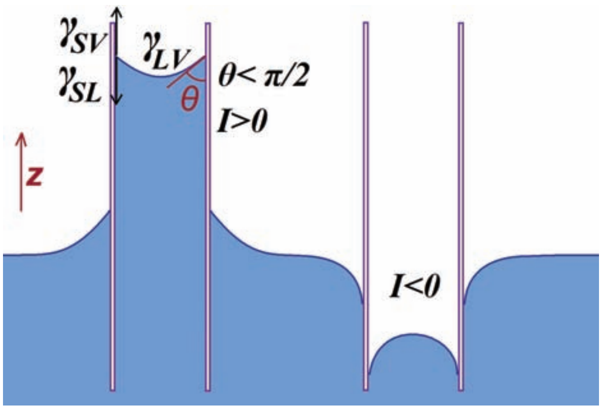

Figure 8.1: Capillary rise and fall in a tube for two values of the imbibition parameter I:
$I > 0$ (*left*) and $I < 0$ (*right*).

We want to predict the dependence of rise height H on both tube radius a and wetting properties. We do so by minimizing the total system energy, specifically the surface and gravitational potential energies. The energy of the water column:

$$E = \underbrace{(\gamma_{SL} - \gamma_{SV})\, 2\pi a H}_{surface\ energy} + \underbrace{\frac{1}{2}\rho g a^2 \pi H^2}_{grav.P.E.} = -2\pi a H I + \frac{1}{2}\rho g a^2 \pi H^2$$

will be a minimum with respect to H when $\frac{\mathrm{d}E}{\mathrm{d}H} = 0$
$\Rightarrow H = 2\frac{\gamma_{SV}-\gamma_{SL}}{\rho g a} = 2\frac{I}{\rho g a}$, from which we deduce

$$\texttt{Jurin's Law} \qquad H = 2\frac{\gamma_{LV}\cos\theta}{\rho g r} \tag{8.1}$$

Note:

1. describes both capillary rise and descent: sign of H depends on whether $\theta > \pi/2$ or $\theta < \pi/2$
2. H increases as θ decreases. H_{max} for $\theta = 0$
3. we've implicitly assumed $R \ll H$ & $R \ll l_C$.

The same result may be deduced via pressure or force arguments.

By Pressure Argument

Provided $a \ll \ell_c$, the meniscus will take the form of a spherical cap with radius $R = \frac{a}{\cos\theta}$. Therefore
$p_A = p_B - \frac{2\sigma\cos\theta}{a} = p_0 - \frac{2\sigma\cos\theta}{a} = p_0 - \rho g H$
$\Rightarrow H = \frac{2\sigma\cos\theta}{\rho g a}$ as previously.

By Force Argument

The weight of the column supported by the tensile force acting along the contact line:
$\rho\pi a^2 H g = 2\pi a(\gamma_{SV} - \gamma_{SL}) = 2\pi a\sigma\cos\theta$, from which Jurin's Law again follows.

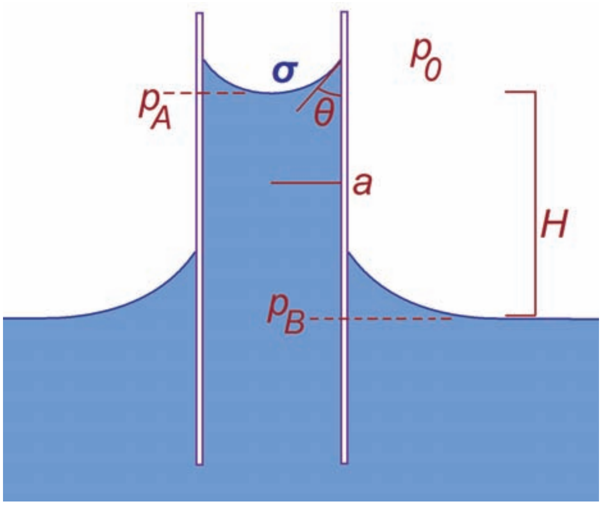

Figure 8.2: Deriving the height of capillary rise in a tube via pressure arguments.

8.1 Dynamics

The column rises due to capillary forces, its rise being resisted by a combination of gravity, viscosity, fluid inertia and dynamic pressure. Conservation of momentum dictates $\frac{\mathrm{d}}{\mathrm{d}t}\left(m(t)\dot{z}(t)\right) = F_{TOT} + \int_S \rho \boldsymbol{v}\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}A$, where the second term on the right-hand side is the total momentum flux, which evaluates to $\pi a^2\rho\dot{z}^2 = \dot{m}\dot{z}$, so the force balance on the column may be expressed as

$$\Big(\underbrace{m}_{Inertia} + \underbrace{m_a}_{Added\ mass}\Big)\ddot{z} = \underbrace{2\pi a\sigma\cos\theta}_{capillary\ force} - \underbrace{mg}_{weight} - \underbrace{\pi a^2\frac{1}{2}\rho\dot{z}^2}_{dynamic\ pressure} - \underbrace{2\pi a z\cdot\tau_v}_{viscous\ force} \tag{8.2}$$

where $m = \pi a^2 z\rho$. Now assume the flow in the tube is fully developed Poiseuille flow, which will be established after a diffusion time $\tau = \frac{a^2}{\nu}$. Thus, $u(r) = 2\dot{z}\left(1 - \frac{r^2}{a^2}\right)$, and $F = \pi a^2\dot{z}$ is the flux along the tube.
The stress along the outer wall: $\tau_\nu = \mu\frac{du}{dr}|_{r=a} = -\frac{4\mu}{a}\dot{z}$.
Finally, we need to estimate m_a, which will dominate the dynamics at short time. We thus estimate the change in kinetic energy as the column rises from z to $z+\Delta z$. $\Delta E_k = \Delta\left(\frac{1}{2}mU^2\right)$, where $m = m_c + m_0 + m_\infty$ (mass in the column, in the spherical cap, and all the other mass, respectively). In the column, $m_c = \pi a^2 z\rho$, $u = U$. In the spherical cap, $m_0 = \frac{2\pi}{3}a^3\rho$, $u = U$. In the outer region, radial inflow extends to ∞, but $u(r)$ decays.
Volume conservation requires: $\pi a^2 U = 2\pi a^2 u_r(a) \Rightarrow u_r(a) = U/2$.
Continuity thus gives: $2\pi a^2 u_r(a) = 2\pi r^2 u_r(r) \Rightarrow u_r(r) = \frac{a^2}{r^2}u_r(a) = \frac{a^2}{2r^2}U$.
Thus, the K.E. in the far field: $\frac{1}{2}m_\infty^{eff}U^2 = \frac{1}{2}\int_a^\infty u_r(r)^2\ dm$, where $dm = \rho 2\pi r^2 dr$.

Hence

$$m_\infty^{eff} = \frac{1}{U^2}\int_a^\infty \rho\left(\frac{a^2}{2r^2}U\right)^2 2\pi r^2 dr = $$
$$= \pi\rho a^4\int_a^\infty \frac{1}{2r^2}dr = \frac{1}{2}\rho\pi a^3$$

Now

$$\Delta E_k = \frac{1}{2}\Delta\left(m_c + m_0 + m_\infty\right)U^2 + \frac{1}{2}m2U\Delta U =$$
$$= \frac{1}{2}\Delta m_c U^2 + \frac{1}{2}\left(m_c + m_0 + m_\infty^{eff}\right)2U\Delta U =$$
$$= \tfrac{1}{2}\left(\pi a^2\rho\Delta z\right)U^2 + \left(\pi a^2\rho z + \tfrac{2}{3}\pi a^3\rho + \tfrac{1}{2}\pi a^3\rho\right)U\Delta U$$

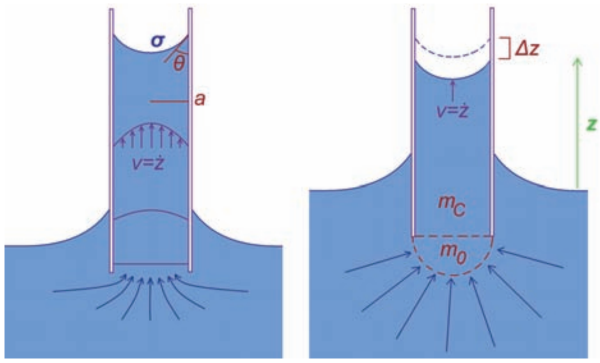

Figure 8.3: The dynamics of capillary rise.

Substituting for $m = \pi a^2 z\rho$, $m_a = \frac{7}{6}\pi a^3\rho$ (added mass) and $\tau_v = -\frac{4\mu}{a}\dot{z}$ into (8.2) we arrive at

$$\left(z + \frac{7}{6}a\right)\ddot{z} = \frac{2\sigma\cos\theta}{\rho a} - \frac{1}{2}\dot{z}^2 - \frac{8\mu z\dot{z}}{\rho a^2} - gz \tag{8.3}$$

The static balance clearly yields the rise height, i.e. Jurin's Law. But how do we get there?

Inertial Regime

1. the timescale of establishment of Poiseuille flow is $\tau^* = \frac{4a^2}{\nu}$, the time required for boundary effects to diffuse across the tube

2. until this time, viscous effects are negligible and the capillary rise is resisted primarily by fluid inertia

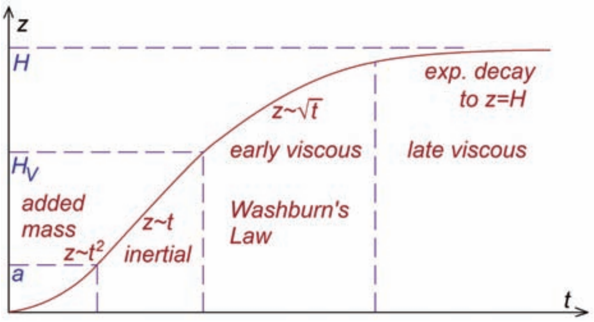

Figure 8.4: The various scaling regimes of capillary rise.

Initial Regime: $z \sim 0$, $\dot{z} \sim 0$, so the force balance assumes the form $\frac{7}{6}a\ddot{z} = \frac{2\sigma\cos\theta}{\rho a}$ We thus infer $z(t) = \frac{6}{7}\frac{\sigma\cos\theta}{\rho a^2}t^2$.
Once $z \geq \frac{7}{6}a$, one must also consider the column mass, and so solve $\left(z + \frac{7}{6}a\right)\ddot{z} = \frac{2\sigma\cos\theta}{\rho a}$. As the column accelerates from $\dot{z} = 0$, $\dot{z}^2$ becomes important, and the force balance becomes: $\frac{1}{2}\dot{z}^2 = \frac{2\sigma\cos\theta}{\rho a} \Rightarrow$
$\dot{z} = U = \left(\frac{4\sigma\cos\theta}{\rho a}\right)^{1/2}$ is independent of g, μ.
$z = \left(\frac{4\sigma\cos\theta}{\rho a}\right)^{1/2} t$.

Viscous Regime ($t \gg \tau^*$) Here, inertial effects become negligible, so the force balance assumes the form: $\frac{2\sigma\cos\theta}{\rho a} - \frac{8\mu z\dot{z}}{\rho a^2} - gz = 0$. We thus infer $H - z = \frac{8\mu z\dot{z}}{\rho g a^2}$, where $H = \frac{2\sigma\cos\theta}{\rho g a}$, $\dot{z} = \frac{\rho g a^2}{8\mu}\left(\frac{H}{z} - 1\right)$
Nondimensionalizing: $z^* = z/H$, $t^* = t/\tau$, $\tau = \frac{8\mu H}{\rho g a^2}$;
We thus have $\dot{z}^* = \frac{1}{z^*-1} \Rightarrow dt^* = \frac{z^*}{1-z^*}dz^* = (-1 - \frac{1}{1-z^*})dz^* \Rightarrow t^* = -z^* - \ln(1 - z^*)$.
`Note`: at $t^* \to \infty$, $z^* \to 1$.

Early Viscous Regime: When $z^* \ll 1$, we consider $\ln(z^* - 1) = -z^* - \frac{1}{2}z^{*2}$and so infer $z^* = \sqrt{2t^*}$.
Redimensionalizing thus yields *Washburn's Law*: $z = \left[\frac{\sigma a\cos\theta}{2\mu}t\right]^{1/2}$
`Note` that $\dot{z}$ is independent of g.

Late Viscous Regime: As z approaches H, $z^* \approx 1$. Thus, we consider $t^* = [-z^* - \ln(1-z^*)] = \ln(1-z^*)$ and so infer $z^* = 1 - \exp(-t^*)$.
Redimensionalizing yields $z = H\left[1 - \exp(-t/\tau)\right]$, where $H = \frac{2\sigma\cos\theta}{\rho g a}$ and $\tau = \frac{8\mu H}{\rho g a^2}$.
`Note`: if rise timescale $\ll \tau^* = \frac{4a^2}{\nu}$, inertia dominates, i.e. $H \ll U_{intertial}\tau^* = \left(\frac{4\sigma\cos\theta}{\rho a}\right)^{1/2}\frac{4a^2}{\nu} \Rightarrow$ inertial overshoot arises, giving rise to oscillations of the water column about its equilibrium height H.

Wicking In the viscous regime, we have $\frac{2\sigma\cos\theta}{\rho a} = \frac{8\mu z\dot{z}}{\rho a^2} + \rho g$. What if the viscous stresses dominate gravity? This may arise, for example, for predominantly horizontal flow (Fig. 8.5).
Force balance: $\frac{2\sigma a\cos\theta}{8\mu} = z\dot{z} = \frac{1}{2}\frac{d}{dt}z^2 \Rightarrow z = \left(\frac{\sigma a\cos\theta}{2\mu}t\right)^{1/2} \sim \sqrt{t}$ (`Washburn's Law`).
`Note`: Front slows down, not due to g, but owing to increasing viscous dissipation with increasing column length.

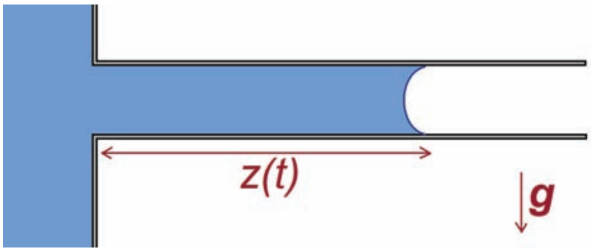

Figure 8.5: Horizontal flow in a small tube.

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