

### Problem Set 3

Due at lecture on Th Oct 26.

1. **Inelastic Diffusion.** Consider a ball bouncing on a rough surface. Each time the ball hits the surface it is scattered in a random direction. For any real surface, the collision is *inelastic*, i.e. the ball only retains a fraction  $0 < r < 1$  of its kinetic energy ( $r =$  “the coefficient of restitution”). Therefore, the ball’s expected height and horizontal displacement are reduced by factors of  $r$  and  $\sqrt{r}$ , respectively, with each successive bounce.

A reasonable model for this situation might be an ‘inelastic random walk’, with exponentially decreasing step lengths<sup>1</sup>. Let  $\Delta X_n$  be IID random variables with zero mean and cumulants  $c_l < \infty$  ( $l \geq 2$ ), which represent the typical displacement after an elastic bounce. The inelastic nature of the collisions is reflected in a rescaling of this distribution with each step. Specifically, our model is the random walk

$$X_N = \sum_{n=1}^N a^n \Delta X_n$$

with non-identical steps, where  $0 < a < 1$  is a constant ( $a = \sqrt{r}$ ). Do the analysis below for the case of one dimension (which would model transverse diffusion on a surface with random parallel grooves), but keep in mind that your results are easily generalized to higher dimensions.

- Express the PDF,  $P_N(x)$ , of  $X_N$  in terms of the PDF,  $p(x)$ , of  $\Delta X_n$ .
- Find the cumulants  $C_{N,l}$  of  $X_N$  (in terms of  $c_l$ ).
- Let  $C_l = \lim_{N \rightarrow \infty} C_{N,l}$  and  $a = 1 - \epsilon$  ( $\epsilon > 0$ ). Show that  $C_{2m}/C_2^m = O(\epsilon^{m-1})$  as  $\epsilon \rightarrow 0$ .
- Let  $\phi(\zeta, \epsilon) = C_2^{1/2} P_\infty(\zeta C_2^{1/2})$ , and show that “the Central Limit Theorem holds” as  $a \rightarrow 1$ . In other words, show that

$$\phi(\zeta, \epsilon) \rightarrow \phi_o(\zeta) = e^{-\zeta^2/2}/\sqrt{2\pi}$$

as  $\epsilon \rightarrow 0$  with  $\zeta$  fixed<sup>2</sup>. This, of course, agrees with the limit of a simple random walk ( $a = 1$ ).

2. **Breakdown of the CLT for decaying walks.** Simulate the two dimensional random walk where

$$p_n(x, y) = \frac{1}{4} (\delta_{y,0}(\delta_{x,a^n} + \delta_{x,-a^n}) + \delta_{x,0}(\delta_{y,a^n} + \delta_{y,-a^n})) \quad (1)$$

is the PDF of the  $n$ th (independent) step, which moves in the square lattice directions but has decaying step size  $a^n$ , as in problem 1. Plot the final positions  $\vec{X}_\infty$  (simulated as  $N \gg 1$ ) of a large number of walks for each of the choices of  $a$  below.

- Compare the PDF of  $X_\infty$  ( $x$  component) for  $a = 0.99$  to the approximation of Problem 1(d).
- What is the exact PDF of  $\vec{X}_\infty$  for  $a = 1/2$ ?
- Describe the CDF (cumulative distribution function) of  $\vec{X}_\infty$  for  $a = 1/3$ .
- Explore other values through simulations, such as  $a = 2^{-1/2}, 2^{-1/4}, 2^{-1/8}, \dots$  and  $a = (\sqrt{5} - 1)/2$  (the golden mean).

<sup>1</sup>See Lecture 14, 18.366 notes (2003).

<sup>2</sup>Note, however, that the CLT does not apply for any fixed  $\epsilon < 0$  as  $N \rightarrow \infty$ .

3. **Shock structure.** Find the traveling wave solution to Burgers equation,

$$c_t + cc_x = Dc_{xx}$$

with boundary conditions  $c(-\infty, t) = c_-$  and  $c(\infty, t) = c_+ < c_-$ , which describes a shock moving at velocity  $v(c_+, c_-)$ .

4. **Discrete vs. continuous models with nonlinear drift.** From problem 3, we find that the continuous traffic flow problem

$$\rho_t + (u(\rho)\rho)_x = D\rho_{xx} \quad (2)$$

$$u(\rho) = u_{max}(1 - \rho/\rho_{max}) \quad (3)$$

$$\rho(-\infty, t) = \frac{1}{4}\rho_{max} \quad \rho(\infty, t) = \frac{3}{4}\rho_{max} \quad (4)$$

has a solution for the form

$$\rho(x, t) = \frac{1}{2}\rho_{max} \left[ 1 + \frac{1}{2} \tanh\left(\frac{x}{\lambda}\right) \right], \quad \lambda = \frac{2D}{u_{max}}. \quad (5)$$

which describes a stationary shock at the origin of width  $\lambda$ .

Following the guidelines below, perform simulations of two discrete models (“A” and “B”), which would seem to be described by the continuum approximation (2). Compare the mean concentration profile  $\bar{\rho}(x)$  averaged in time (after initial transients, assuming they decay) to the continuum approximation (5). Also plot some particle trajectories in the  $(x, t)$  plane for both models.

Assume  $u_{max} = 1$ ,  $\lambda = 1$ ,  $D = 1/2$ , and  $\rho_{max} = 100$ . It would also be interesting (but not required) to vary these parameters, and the simulation algorithms below.

- *Initial condition.* The simulation region is  $[L, L]$ , where  $L \gg \lambda$  is needed to resolve the shock profile. Initially, particles can be placed uniformly at random in  $[-L, 0]$  with concentration  $\rho_{max}/4$ , and in  $[0, L]$  with concentration  $3\rho_{max}/4$ . For example with  $L = 4$ , place 100 particles in  $[-4, 0]$  and 300 particles in  $[0, 4]$ .
- *Local concentration.* In the models below, a particle at  $x$  interacts with its neighbors by responding to the local concentration  $\tilde{\rho}(x)$  and local gradient  $\tilde{\rho}_x(x)$ . Measure these quantities over a scale  $l$ , which should be larger than the mean spacing on the left  $l \gg 4/\rho_{max}$  and smaller than the scale of large macroscopic variations,  $l < \lambda$  (e.g.  $l = 0.2$ ). For example,  $\tilde{\rho}(x)$  could be the number of particles (including itself) in the region  $[x - l, x + l]$  divided by  $2l$ , and  $\tilde{\rho}_x(x)$  could be the number on the right minus the number on the left divided by  $l$  (with appropriate changes near boundaries at  $x = \pm L$ ).
- *Left boundary condition.* Since particles near the left boundary have a drift velocity to the right  $u(\rho_{max}/4) = 3u_{max}/4$  and concentration  $\rho_{max}/4$ , we can approximate (4) by introducing a new particle at  $x = -L$  after each time step  $\tau$  with probability  $\tau/\bar{\tau}$  ( $\tau < \bar{\tau}$ ) where  $\bar{\tau} = 16/(3\rho_{max}u_{max})$ . (Explain why.) If a particle crosses the left boundary  $x = -L$ , it should be reflected back into the simulation region to preserve the desired mean concentration.
- *Right boundary condition.* Particles tend to leave the simulation region on the right at  $x = L$ , but their density must be controlled to set  $\rho = 3\rho_{max}/4$ . One way to do this in each time step could be to remove all particles with  $x > L - l$ , and replace them with uniformly distributed particles in  $[L - l, L]$  with the desired concentration.
- *Dynamics.* We consider two completely different models, which might be described by (2).

- (a) (Discrete Model A, “gas dynamics”) Since Equation (2) resembles the Fokker-Planck equation with a concentration-dependent drift,  $u(\rho)$ , and diffusivity  $D$ , in each time step  $\tau$ , give each particle a random displacement

$$\begin{aligned}\Delta x &= u(\tilde{\rho})\tau \pm \sqrt{2D\tau} \\ &= u_{max}\tau \left(1 - \frac{\tilde{\rho}}{\rho_{max}}\right) \pm \sqrt{\lambda u_{max}\tau}\end{aligned}$$

where  $+$  and  $-$  are chosen with equal probability  $1/2$ . So as not to invalidate the local density measurement, the time step should be chosen such that  $u_{max}\tau \ll l$ , e.g.  $\tau = 0.01$ . Note that the random diffusive displacements are much larger than the deterministic drift displacements (and thus often backwards), as in the derivation of the Fokker-Planck equation for a Wiener process. This kind of model resembles gas dynamics (for the velocity  $u(\rho)$  at a shock).

- (b) (Discrete Model B, “traffic flow”) In our continuum model for traffic flow, the motivation for the diffusion term is that drivers “look ahead” and adjust their speed according to the local concentration gradient. Therefore, also try a fully deterministic model,

$$\Delta x = \left( u(\tilde{\rho}) - \frac{D}{\tilde{\rho}} \tilde{\rho}_x \right) \tau$$

This model seems very different from Model A and may be unstable and more sensitive to initial conditions and parameters, but chaos in other deterministic nonlinear systems is known to lead to simple diffusive behavior, which resembles a random process.