18.366 Random Walks and Diffusion, Fall 2006, M. Z. Bazant.

Problem Set 4

Due at lecture on Th Nov 9.

1. **First passage for biased diffusion.** Consider a continuous diffusion process with drift velocity v and diffusivity D which starts at $x_0 > 0$ at $t = 0$.

   (a) Derive the PDF $f(t)$ of the first passage time to the origin. Plot the result for $v = 0, 1, -1$.

   (b) Derive the survival probability $S(t)$. For $v > 0$, what is the probability of eventual first passage?

   (c) What is the PDF of the minimum first passage time $T_{min}(N)$ of a set of N independent processes of this type?

2. **First passage for anomalous walks.** Simulate a Cauchy walk starting at $x = 1$ with displacement PDF

$$p(x) = \frac{1}{\pi((x-d)^2+1)}.$$

   (a) For the unbiased case ($d = 0$), compute the distrbution $f(n)$ of the first passage time to the origin, i.e. probability that the walk first crosses to $x < 0$ in step n. How does $f(n)$ behave for large n? [Compare to the decay $f(n) \propto n^{-3/2}$ obtained in class for normal diffusion (Smirnov density).] Estimate the probability the walker eventually crosses the origin.

   (b) Repeat part (a) for the cases $d = 1$ and $d = -1$ of bias away from or toward the origin, respectively. Compare to the results of problem 1.

3. **First passage to a sphere.** Consider an unbiased continuous diffusion process starting at a distance $r_0 > R$ away from the center of a sphere of radius R. Derive the ratio of the eventual hitting PDF at the nearest point to the eventual hitting PDF at the farthest point on the sphere.

4. **The Ballot Problem.**

   (a) In a two-person election, candidate A receives p votes, and candidate B receives q votes, where $q \leq p$, and all the votes are placed in a ballot box. To count the votes, they are randomly removed one at a time from the box. Show that the probability of A always having strictly more votes than B during the whole counting process is $(p - q)/(p + q)$. You may find it helpful to consider the difference between the two partial scores as a random walk.

   (b) Now consider an election with an additional candidate C who achieves r votes, where $r \leq p$. Write a code to simulate the counting process of the three candidates, and numerically find the probability $P(p, q, r)$ that candidate A always has strictly more votes than candidates B and C. Consider cases where p, q and r are smaller than 10.

   (c) *Extra credit.* From part A, we know that $P(p, q, 0) = (p - q)/(p + q)$. Derive an exact expression for $P(p, q, 1)$. What about a general $P(p, q, r)$?