

Lecture 13: Discrete and Continuous Stochastic Processes

Scribe: Sergiy Sidenko

Department of Mathematics, MIT

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Discrete Markov Processes

In previous lectures we considered random walks with independent steps which were indifferent to the current position of the particle. Now we are going to use the conditioned transition probabilities $p_N(\bar{x}|\bar{x}')$ instead of $p_N(\bar{x})$, where \bar{x}' stands for the position after N steps. Such random walks are called *Markovian random walks*.

Chapman-Kolmogorov Equation

Clearly, the probability density function $P_{N+1}(\bar{x})$ after $N + 1$ steps can be obtained recursively through the following equation which can be thought as a generalization of the Bachelier equation:

$$P_{N+1}(\bar{x}) = \int p_N(\bar{x}|\bar{x}') P_{N+1}(\bar{x}') d\bar{x}'. \quad (1)$$

For the sake of simplicity, we consider Markov processes in 1-dimension space; however, all the following equations can be derived for random walks in \mathbb{R}^n .

Let us consider a more general function $\rho(x, t)$ such that it satisfies the identity $\rho(x, N\tau) = P_N(x)$ for all nonnegative integers N . Here, τ denotes a small period of time. Now, introducing a continuous time $t = N\tau$ and a new transition probability function $p(x, t)$ as

$$p(x, t + \tau|x', t) = p_N(x|x'),$$

we can restate the equation (1) as follows:

$$\rho(x, t + \tau) = \int p(x, t + \tau|x', t) \rho(x', t) dx'. \quad (2)$$

The last formula is called the *Chapman-Kolmogorov equation*.

We can compute the (centered) moments of transitions as usual:

$$M_n(x', t) = \langle (x - x')^n \rangle = \int p(x, t + \tau|x', t) (x - x')^n dx. \quad (3)$$

Kramers-Moyall Expansion

In this section we will obtain *formally* a PDE for the function $\rho(x, t)$ as N tends to infinity. Let us introduce a new variable $y = x - x'$. Then, (2) can be rewritten as follows:

$$\rho(x, t + \tau) = \int p(x + y - y, t + \tau | x - y, t) \rho(x - y, t) dy.$$

Now assuming that all moments $M_n(x', t)$ are finite for all x and t , we expand the right-hand side (here we use the fact that $M_0 \equiv 1$) and integrate each term separately:

$$\begin{aligned} \rho(x, t + \tau) &= \int \sum_{n=0}^{n=\infty} \frac{(-y)^n}{n!} \frac{\partial^n}{\partial x^n} [p(x + y, t + \tau | x, t) \rho(x, t)] dy \\ &= \sum_{n=0}^{n=\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} [M_n(x, t) \rho(x, t)]. \end{aligned}$$

Since the first term in the right-hand side is merely $\rho(x, t)$, we can rewrite the last formula as follows:

$$\frac{\rho(x, t + \tau) - \rho(x, t)}{\tau} = \frac{\partial \rho}{\partial t} + \sum_{n=2}^{n=\infty} \frac{\tau^{n-1}}{n!} \frac{\partial^n \rho}{\partial t^n} \quad (4)$$

$$= \sum_{n=1}^{n=\infty} (-1)^n \frac{\partial^n}{\partial x^n} [D_n(x, t) \rho(x, t)], \quad (5)$$

where $D_n(x, t) = \frac{M_n(x, t)}{n! \tau}$.

Note that we can apply this approach only in the ‘central region’, but not in ‘tails’ of the Green function $G(x, t | x_0, 0)$ which solves the PDE with the initial condition $\rho(x, 0) = \delta(x - x_0)$.

Now we are able to do two things: either to get (by recursive substitution) a valid asymptotic expansion of $\rho(x, t)$ as N tends to infinity with t fixed, or to consider the limit with $\tau \rightarrow 0$ assuming some scaling for $M_n(x, t)$. It turns out that these two methods are not equivalent.

Modified Kramers-Moyall Expansion

In the first approach mentioned above we have:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\frac{\partial}{\partial x} (D_1 \rho) + \frac{\partial^2}{\partial x^2} (D_2 \rho) - \frac{\partial^3}{\partial x^3} (D_3 \rho) + \frac{\partial^4}{\partial x^4} (D_4 \rho) \\ &+ \frac{\tau}{2} \frac{\partial}{\partial x} \left(\rho \frac{\partial D_1}{\partial t} + D_1 \left[-\frac{\partial}{\partial x} (D_1 \rho) + \frac{\partial^2}{\partial x^2} (D_2 \rho) \right] \right) \\ &- \frac{\tau}{2} \frac{\partial^2}{\partial x^2} \left(\rho \frac{\partial D_2}{\partial t} + D_2 \left[-\frac{\partial}{\partial x} (D_1 \rho) + \frac{\partial^2}{\partial x^2} (D_2 \rho) \right] \right) + \dots \end{aligned}$$

Now assuming that all D_n are constants, we arrange all terms depending on the orders of partial space derivatives:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x} (D_1 \rho) + \frac{\partial^2}{\partial x^2} (D_2 \rho) - \frac{\tau}{2} \frac{\partial}{\partial x} \left(D_1 \frac{\partial}{\partial x} (D_1 \rho) \right) + \dots,$$

which can be rewritten as follows:

$$\frac{\partial \rho}{\partial t} = \left(D_2 - D_1^2 \frac{\tau}{2} \right) \frac{\partial^2 \rho}{\partial x^2} = \frac{\sigma^2}{2\tau} \frac{\partial^2 \rho}{\partial x^2},$$

where $\sigma^2 = M_2 - M_1^2$ is the variance of the displacements, not the second moment, as expected from the Central Limit Theorem.

If we want to consider the limit with $\tau \rightarrow 0$, we clearly should omit the terms with τ .

Continuous Stochastic Processes

Fokker-Planck Equation

Let us now consider the second approach, namely, $\tau \rightarrow 0$. Assume that all moments scale (for all x and t) as in the Central Limit Theorem:

$$\begin{aligned} M_1, M_2 &= O(\tau), \\ M_n &= O\left(\tau^{n/2}\right) \quad \text{if } n > 2. \end{aligned}$$

With these scalings, the expansion (4) is asymptotic to $O(\tau)$ and the leading order equation gives us the *Fokker-Planck equation*:

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial}{\partial x} (D_1 \rho) = \frac{\partial^2}{\partial x^2} (D_2 \rho), \quad (6)$$

where

$$\begin{aligned} D_1(x, t) &= \lim_{\tau \rightarrow 0} \frac{M_1(x, t)}{\tau}, \\ D_2(x, t) &= \lim_{\tau \rightarrow 0} \frac{M_2(x, t)}{2\tau}. \end{aligned}$$

Stochastic Differential Equations

Let us consider the following continuous-time stochastic process:

$$x_t = \int_0^t dx_t, \quad (7)$$

where

$$dx_t = a(x_t, t) dt + b(x_t, t) dz. \quad (8)$$

In (8) $a(x_t, t)$ denotes the mean drift of the process, $b(x_t, t)$ represents the variant rate and dz stands for so-called *Wiener's Stochastic differential*. The integral of dz_t

$$z_t = \int dz_t \quad (9)$$

generates *Wiener process*, or *Brownian motion*. By assumption, we have the following properties of dz_t :

$$\begin{aligned} \langle dz_t \rangle &= 0, \\ \langle dz_t^2 \rangle &= dt. \end{aligned}$$

One (perhaps, the most intuitive) way of thinking about dz_t is to consider it as an infinitesimal Gaussian displacement.

As we see, by construction the probability density function of x_t exactly satisfies the Fokker-Planck equation with

$$\begin{aligned} D_1(x, t) &= a(x, t), \\ D_2(x, t) &= \frac{b^2(x, t)}{2}. \end{aligned}$$

Itô calculus

It turns out that the ordinary rules for differentiation are not applicable if we want to find the derivative of $f(x_t, t)$, where x_t is defined as in (7). The famous *Itô's Lemma* says that in fact the differential of f can be found as follows:

$$df = \left(\frac{\partial f}{\partial t} + a \frac{\partial f}{\partial x} + \frac{b^2}{2} \frac{\partial^2 f}{\partial x^2} \right) dt + \left(b \frac{\partial f}{\partial x} \right) dz. \quad (10)$$

The intuition behind the additional terms is that dx_t is actually of order \sqrt{dt} ; thus, we must carry extra terms to keep the right-hand side of order dt .

Application in Finance

In the 'Black-Scholes world' we might have the following 'geometric Brownian motion' (or *lognormal process*)

$$dx_t = \mu x dt + \sigma x dz = x(\mu dt + \sigma dz),$$

where σ is the volatility and μ is the expected rate of return. Thus, the corresponding process x_t depends only on two quantities: μ and σ .

If we consider, for instance, the function $y = \log x$, then the Itô's lemma gives us for dy :

$$dy = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dz = \bar{\mu} dt + \sigma dz.$$

Therefore, drift in $x = e^y$ is $(\bar{\mu} + \frac{\sigma^2}{2})x$, and:

$$\left\langle \frac{\Delta x}{x} \right\rangle = \bar{\mu} + \frac{\sigma^2}{2},$$

which corresponds to the earlier obtained formulas.

Let us now apply Itô's lemma to the hedged portfolio

$$u = w - \phi x, \quad (11)$$

where w is the option price and $\phi = \frac{\partial w}{\partial x}$ is the hedge ratio. Formula (10) gives us:

$$\begin{aligned} du &= dw - \phi dx - x d\phi - dx d\phi \\ &= dw - \phi dx - (x + dx) d\phi. \end{aligned}$$

The last term here represents investing money right after the time step at t , because $x + dx$ is a new price and $d\phi$ is a change in the asset. Since in practice nobody wants to be bound to

invest money in the same asset (we might want to do something else with this money), the term $(x + dx) d\phi$ is neglected. Another reason for this is that nobody wants to link future investments to the profit gained at time t .

After that we have:

$$\begin{aligned} du &= dw - \phi dx \\ &= \left(\frac{\partial w}{\partial t} + \mu x \frac{\partial w}{\partial x} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 w}{\partial x^2} \right) dt + \left(\sigma x \frac{\partial w}{\partial x} \right) dz - \left(\frac{\partial w}{\partial x} \right) (\mu x dt + \sigma x dz). \end{aligned}$$

Here the term with dz drops out and we have no risk dependence in the equation. Also, mean return $\mu x \frac{\partial w}{\partial x}$ is canceled, and we obtained

$$du = \left(\frac{\partial w}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 w}{\partial x^2} \right) dt.$$

Now we want u to grow at rate r which is the *risk free rate*. Thus, we must claim that

$$du = r u dt.$$

Substituting $w - \phi x$ for u and $\frac{\partial w}{\partial x}$ for ϕ , we get the *Black-Scholes equation*:

$$\begin{aligned} \left(\frac{\partial w}{\partial t} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 w}{\partial x^2} \right) dt &= r \left(w - x \frac{\partial w}{\partial x} \right) dt \\ \frac{\partial w}{\partial t} + r x \frac{\partial w}{\partial x} + \frac{\sigma^2 x^2}{2} \frac{\partial^2 w}{\partial x^2} &= r w. \end{aligned} \tag{12}$$