Lecture 13: Extereme Events, Lévy Stability and Fractional Calculus

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1 Extreme Events

Consider N i.i.d. random variables with PDF $p(x)$, i. e. an N-step random walk. The question we ask now is: How is the largest step up to time N distributed? Here we focus on fat-tail random variables with PDF

$$p(x) \sim \frac{A}{x^{1+\alpha}} \qquad \text{as } x \to \infty. \tag{1}$$

For example, we may take $p(x) = l_\alpha(x)$.

The outcomes can now be ordered

$$\Delta x_{(1)} \leq \Delta x_{(12)} \leq \ldots \leq \Delta x_{(N)}, \tag{2}$$

so that the largest value cumulative distribution function is defined as

$$F_N(x) \equiv Prob\ (\Delta x_{(N)} \leq x). \tag{3}$$

If the number of steps N is large enough, the largest outcome is always sampled in the tail, where the CDF is

$$P(x) \simeq 1 - \frac{A}{\alpha x^\alpha}. \tag{4}$$

Then, for the largest step we obtain

$$F_N(x) = [P(x)]^N. \tag{5}$$

This equation merely states the simple fact that the largest step is smaller than x if and only if *all* steps are smaller than x. For $N \to \infty$, and $\Delta x_{(N)} \to \infty$,

$$F_N(x) \simeq \left[1 - \frac{A}{\alpha x^\alpha}\right]^N \simeq \exp\left[-\frac{NA}{\alpha x^\alpha}\right]. \tag{6}$$

The expression in the exponent suggests the following rescaling:

$$z_N = \frac{\Delta x_{(N)}}{(AN/\alpha)^{1/\alpha}}. \tag{7}$$

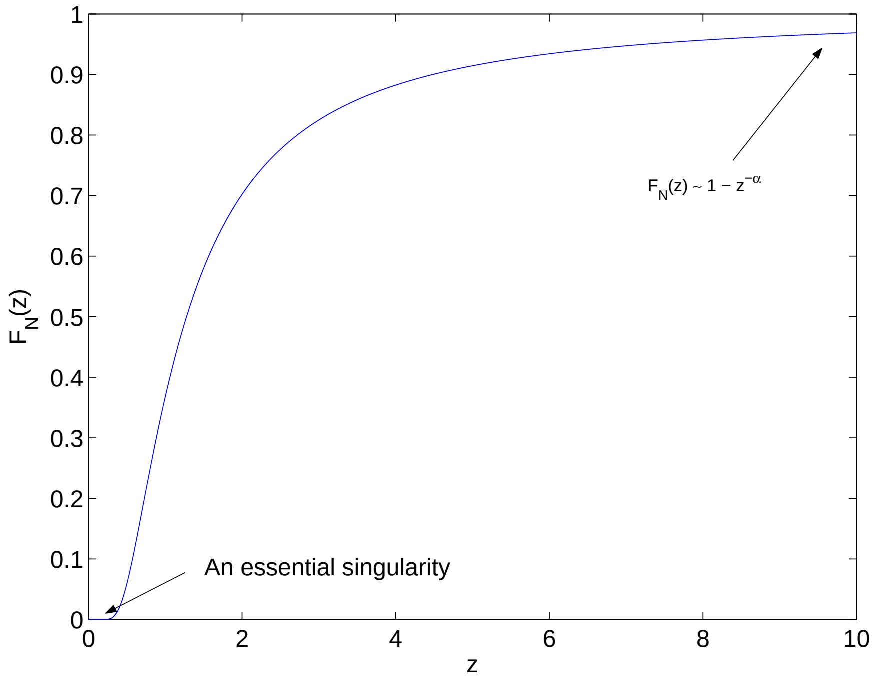

Figure 1: Fréchet distribution is characterized by an essential singularity at the origin and a power-law tail.

Then we rewrite (6) as

$$F_N(z) \sim \exp\left[-z^{-\alpha}\right]. \tag{8}$$

This is called *Fréchet* distribution[1] (introduced in 1926).

Recall now that in a Lévy flight with $p(x) = l_\alpha(x, a) \propto ax^{-(1+\alpha)}$, the position after N steps is distributed like

$$P_N(x) = \frac{1}{N^{1/\alpha}} l_\alpha\left(\frac{x}{N^{1/\alpha}}, a\right), \tag{9}$$

i. e. it has a characteristic width of $N^{-1/\alpha}$. Behold that the extremal value z_N also scales like $N^{-1/\alpha}$. This interesting phenomenon is a direct consequence of the power-law distribution which has no scale and therefore in a typical Lévy flight, *all* sizes of steps are present. The total displacement is therefore of the same order of magnitude as the largest step in the walk (see Fig. 2).

2 Lévy Stability

What are the possible limiting distributions for a sum of i.i.d. random variables? By now, we know the answer to this question for a quite broad class of distributions, with $var(\Delta x) < \infty$, that converge to a Gaussian in accordance with the Central Limit Theorem. The analog of the CLT for fat-tail distributions are the so-called Lévy Stability Laws, that are the subject of the present section.

[1] A nice summary of extreme values distributions, also called *Fisher-Tippett distributions* can be found online at http://mathworld.wolfram.com/Fisher-TippettDistribution.html

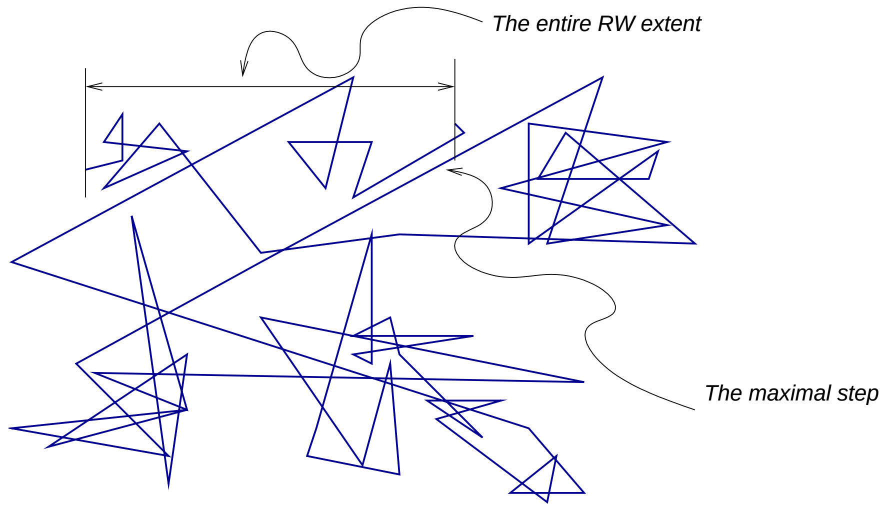

Figure 2: A typical Lévy flight: the maximal step and the entire walk extent are of the same order of magnitude.

For i.i.d. variables,

$$\hat{P}_N(k) = \left[\hat{P}(k)\right]^N. \tag{10}$$

As we have seen in several case studies before, short-time correlations do not influence the long-time behavior of $P_N(x)$; their main effect is the rescaling of the time unit in terms of "correlation time" n_c, so that the sum of N r. v. acts like a sum of N/n_c independent r. v. More generally,

$$\hat{P}_{n\times m}(k) \sim \left[\hat{P}_n(k)\right]^m, \qquad n \gg n_c. \tag{11}$$

For i.i.d. variables, this equation is exact.

In what follows, we assume zero mean for Δx_n. Then, rescaling

$$z_N = \frac{X_N}{a_N}, \tag{12}$$

we find that $F_N(z)$, which is the PDF for z_N, satisfies

$$\hat{F}_{n\times m}(a_{mn}k) = \left[\hat{F}_n(a_n k)\right]^m, \tag{13}$$

or

$$\hat{F}_{n\times m}(\frac{a_{mn}}{a_n}k) = \left[\hat{F}_n(k)\right]^m. \tag{14}$$

Let $\hat{F}_N(k)$ converge to some limit $\hat{F}(k)$ as $N \to \infty$, where $\hat{F}(k)$ is a nontrivial characteristic function. Then

$$\lim_{n\to\infty} \frac{a_{mn}}{a_n} = c_m \qquad (m \text{ is fixed}). \tag{15}$$

Thus, (14) reduces to the following functional equation

$$\hat{F}_m(c_m k) = \left[\hat{F}(k)\right]^m. \tag{16}$$

What are the possible solutions of this equation? To answer this, we use a scaling argument. The scaling constants a_N can be expressed as

$$a_N = N^{1/\alpha} L_N, \tag{17}$$

where α is some constant and L_N is a slowly varying function of N, e. g. $L_N = (\log N)^\mu$.

Proof:

$$\frac{a_{mn}}{a_n} = \frac{(m \times n)^{1/\alpha}}{n^{1/\alpha}} \frac{L_{m\times n}}{L_n} \to c_m = m^{1/\alpha}. \tag{18}$$

Hence, (16) becomes

$$\hat{F}_m(m^{1/\alpha} k) = \left[\hat{F}(k)\right]^m. \tag{19}$$

The only possible solutions of this equation satisfying $\hat{F}(0) = 1$ and $\hat{F}(\infty) = 0$ have the form

$$\hat{F}(k) = \exp\left[-\nu \ \mathrm{sign}(k)|k|^\alpha\right]. \tag{20}$$

If the distribution is symmetric, then

$$\hat{F}(-k) = \hat{F}(k). \tag{21}$$

Since the characteristic function is in this case sign-independent, we obtain

$$\hat{F}(k) = \exp\left[-a|k|^\alpha\right] = \hat{l}_\alpha(a, k), \tag{22}$$

i. e. the Lévy distribution characteristic function. More generally, we have

$$\hat{F}(-k) = \hat{F}^*(k), \tag{23}$$

then rewriting $\nu \ \mathrm{sign}(k) = c_1 + ic_2 \ \mathrm{sign}(k)$ we find that the general form of the limiting distribution reads

$$\hat{F}(k) = \exp\left[-a|k|^\alpha \left(1 - i\beta \tan(\frac{\alpha\pi}{2}) \ \mathrm{sign}(k)\right)\right] \equiv \hat{l}_{\alpha,\beta}(a, k), \tag{24}$$

This is a three-parameter distribution; the parameters $0 < \alpha \leq 2$ and $-1 \leq \beta \leq 1$ determine the shape of the distribution and a determines the width.

The function $\hat{l}_{\alpha,\beta}(a, k)$ thus produces the limiting distribution for a much broader class of random walks.

2.1 Basins of Attractions

In the above context, several results have been obtained regarding the basins of attraction of $l_{\alpha,\beta}(a, x)$ that are worth mentioning here. We state these theorems without a proof.

Theorem 1 (Gnedenko-Doeblin) *The distribution $p(x)$ is the basin of attraction of $l_{\alpha,\beta}(a, x)$, i. e.*

$$\frac{1}{a_N} P_N\left(\frac{x}{a_N}\right) \to l_{\alpha,\beta}(a, x) \tag{25}$$

if and only if

the CDF $P(x) = \int_{-\infty}^{x} p(x')dx'$ satisfies

1.

$$\lim_{x\to\infty} \frac{p(-x)}{1-p(x)} = \frac{1-\beta}{1+\beta} \tag{26}$$

2. $\forall r > 0$,

$$\lim_{x\to\infty} \frac{1-p(x)+p(-x)}{1-p(rx)+p(-rx)} = r^{\alpha} \tag{27}$$

Theorem 2 (Gnedenko) *The distribution* $p(x)$ *is the basin of attraction of* $l_{\alpha,\beta}(a,x)$ *with* $a_N = N^{1/\alpha}$ *if*

$$p(x) \sim \frac{A_{\pm}}{|x|^{1+\alpha}} \qquad x \to \pm\infty, \tag{28}$$

where

$$\beta = \frac{A_+ - A_-}{A_+ + A_-}. \tag{29}$$

3 The Continuum Limit and Fractional Calculus

For a Lévy flight, we recall that

$$\hat{P}_N(k) \sim \exp(-Na|k|^{\alpha}) = \exp\left[-\frac{a}{\tau}|k|^{\alpha} t\right], \tag{30}$$

where, as usual, we define the continuum limit by $t = N\tau$. Then, $\rho(x,t) = P_N(x)$ and the "diffusion equation" in the Fourier space reads

$$\frac{\partial\hat{\rho}}{\partial t} = -\frac{a}{\tau}|k|^{\alpha}\hat{\rho}. \tag{31}$$

The RHS of this equation helps to define the *Riesz fractional derivative* of ρ as the inverse Fourier transform:

$$\frac{\partial^{\alpha}\rho(x)}{\partial|x|^{\alpha}} \equiv \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx}(-|k|^{\alpha})\hat{\rho}(k). \tag{32}$$

This is a formal construction that is more or less frequently used in similar problems.