18.369 Mathematical Methods in Nanophotonics
Spring 2008

18.369 Midterm Exam (Spring 2008)

April 7, 2008

You have two hours. **There are three problems, each worth 30 points.**

Problem 1: Waveguide Gaps

In figure 1(a) is shown a 2d hollow metallic waveguide of width L. If we solve for the 2d TM-polarized (E_z only, z-invariant) eigensolutions in this geometry, they are of the form:

$$E_z(x, y, t) = \sin\left(\frac{\pi n}{L} y\right) e^{i(kx-\omega t)},$$

with n a positive integer and eigenfrequencies (bands) $\omega_n(k) = \sqrt{k^2 + (\pi n/L)^2}$.

[**Useful formulae:** given a set of degenerate eigenmodes $\{\mathbf{E}_\ell\}$ with an unperturbed eigenvalue ω, orthonormalized so that $\langle \mathbf{E}_\ell, \varepsilon \mathbf{E}_m \rangle = \delta_{\ell,m}$, then you should recall that the first-order perturbations $\Delta\omega^{(1)}$ due to a small $\Delta\varepsilon$ are the eigenvalues of the matrix $A_{\ell m} = -\frac{\omega}{2}\langle \mathbf{E}_\ell, \Delta\varepsilon \mathbf{E}_m \rangle$. And the eigenvalues λ of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are, of course, the roots of $\lambda^2 - (a + d)\lambda + (ad - bc) = 0$. Some handy trig. identities: $2\cos^2(u) = 1 + \cos(2u)$, $2\sin^2(u) = 1 - \cos(2u)$, $2\sin(u)\cos(v) = \sin(u + v) + \sin(u - v)$.]

(a) Now, we will take this waveguide and fill it with a *small* periodic (period a) perturbation $\pm\Delta\varepsilon$ as shown in figure 1(b): alternating thickness $a/2$ layers of $\varepsilon = 1 + \Delta\varepsilon$ and $\varepsilon = 1 - \Delta\varepsilon$. **Sketch the band diagram**, assuming $a = L/2$, by starting with the "folded" bands for $n = 1, 2, 3$ (sketched reasonably quantitatively) and then showing qualitatively (no calculations necessary) how they would change for a small $\Delta\varepsilon \approx 0.1$. (What happens when an $n = 1$ and $n = 2$ mode cross? What about $n = 1$ and $n = 3$?)

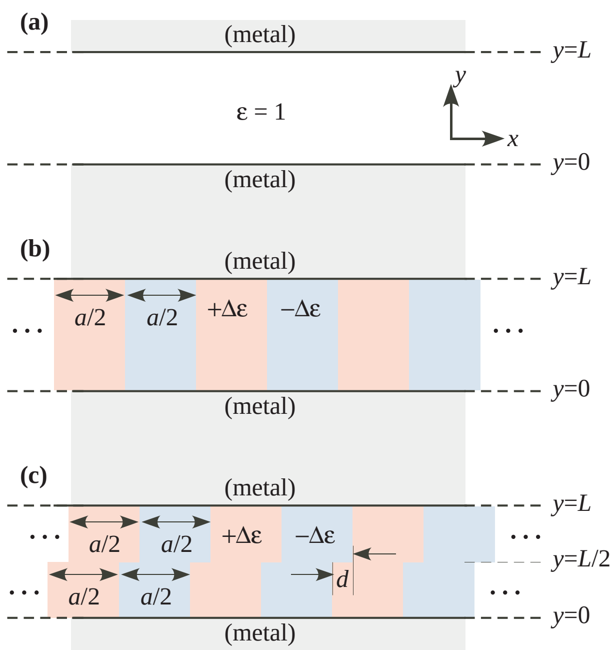

Figure 1: (a) Schematic of a 2d metal waveguide of width L, which supports modes propagating along the x direction. (b) A perturbation $\pm\Delta\varepsilon$ is introduced via two periodic layers of thickness $a/2$ filling the waveguide. (c) The perturbation is modified: for half of the thickness ($y \in [0, L/2]$), the layers are shifted in the x direction by a distance d.

(b) Next, let us further change the perturbation as shown in figure 1(c): for half of the waveguide ($y \in [0, L/2]$), the perturbation is shifted in the x direction by some distance d. Using first-order perturbation theory, **estimate the size of the lowest-ω gap** (to first-order in $\Delta\varepsilon$, as a fraction of mid-gap) that opens at $k = \pi/a$ in the $n = 1$ band for **two cases**: $d = 0$ and $d = a/2$. [Hint: you can use symmetry to eliminate or simplify many of the integrals if you choose your x origin and unperturbed modes appropriately.]

(c) **What is the space group** of the structure in figure 1(c) (including all rotations, mirrors, translations, etc.) for the **two cases** $d = 0$ and $d = a/2$?

Problem 2: Symmetry and Stuff

As shown in figure 2, we arrange N identical masses $m > 0$ onto a circle, uniformly spaced, and attach each to its neighbors by a spring constant $\kappa > 0$. The masses are constrained to move along the circle, and the motion of each mass is described by an angle ϕ_ℓ as shown, where $\phi_\ell = 0$ corresponds to the initial position for mass ℓ.

If we assume a time-dependence $e^{-i\omega t}$ as usual, then the frequencies ω satisfy the eigenproblem $\hat{\Theta}\psi = \omega^2\psi$, where $\psi = (\phi_1, \phi_2, \cdots, \phi_N)^T$ and $\hat{\Theta}$ is the $N \times N$ real-symmetric positive-semi-definite matrix:

$$\hat{\Theta} = \frac{\kappa}{m}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

(a) Obviously, the system in figure 2 is invariant under C_N rotations, corresponding to a *cyclic shift* $\phi_1 \to \phi_2$, $\phi_2 \to \phi_3$, ..., $\phi_{N-1} \to \phi_N$, $\phi_N \to \phi_1$. **Show explicitly** that this $\hat{\Theta}$ **commutes with cyclic shifts**.

(b) Let $D(n)$ be the representation matrix for a rotation C_N^n (i.e. a cyclic shift n times). **What are the possible irreducible representations** for this group (the *cyclic group* of order N)? [Hint: $D(n)D(n') = D(?)$.]

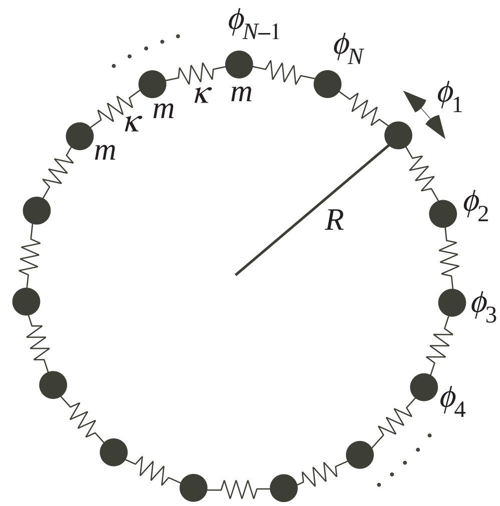

Figure 2: N identical masses m arranged on a circle, connected with spring constants κ, and allowed to slide freely on the circle, where ϕ_ℓ denotes the angular displacement of the ℓ-th mass from its initial position (equally spaced).

(c) Using your answer from (b), **solve for the eigenfrequencies ω and the corresponding eigenvectors**.

(d) Using your answer from (b), **give the projection operator** onto the irreducible representations. Also, **what does this operator become in the limit $N \to \infty$?**

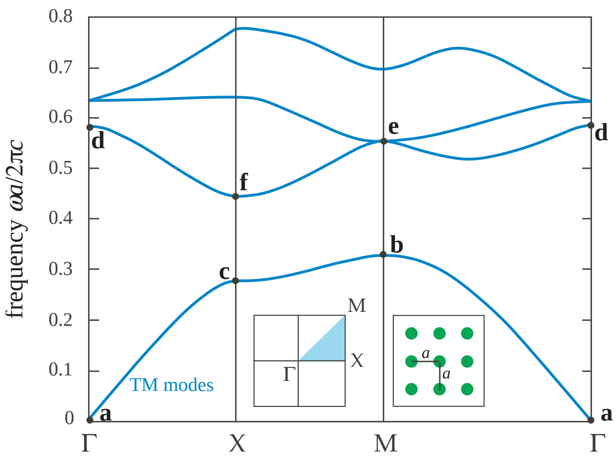

Figure 3: TM band diagram of a square lattice (lattice constant a) of circular dielectric rods (right inset) plotted around the boundary of the irreducible Brillouin zone (left inset). Various points (black dots) are labelled with letters (**a**–**f**) for future reference.

Problem 3: Projected Bands

The TM band diagram of a square lattice (lattice constant a) of circular dielectric rods is shown in figure 3. In class, we considered linear defects along the Γ–X direction (e.g. removing a row of rods). Here, we will consider linear defects along the Γ–M (**diagonal**) direction, with period $a\sqrt{2}$ along that direction.

(a) **Sketch the projected band diagram** along the Γ–M direction: plot **the first two bands** of the periodic crystal as a function of the component k_d of $\mathbf{k}$ along this direction, for the irreducible Brillouin zone in k_d. On your plot, **label** with letters **a**–**f** the points corresponding to those labelled locations in figure 3.

(b) **Sketch (qualitatively)** your best guess for **the projected band diagram** including the modes **of a defect** where N adjacent diagonal rows of rods are removed (e.g. as shown in figure 4 for $N = 3$). **Sketch what happens** as N increases, and in the limit as $N \to \infty$. You may assume that there are *no surface states* for this crystal termination. [Hint: it might be easier to start with the $N \to \infty$ limit and then sketch what happens as N decreases.]

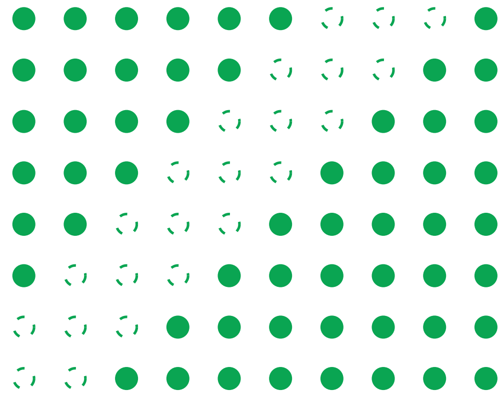

Figure 4: Linear defect in the diagonal (Γ–M) direction of a square lattice of rods formed by removing $N = 3$ adjacent diagonal rows of rods (removed rods shown as dashed outlines).