• Variable Length Pendulum.

Statement:

Consider a pendulum (in a plane), whose arm length $L > 0$ changes in time (i.e.: $L = L(t)$). To make matters more precise:

(a) Let the **hinge** for the pendulum be at origin in the plane: $x = y = 0$.

(b) Let the mass M for the pendulum be at $x = L \sin\theta$ and $y = -L\cos\theta$, where θ is the angle measured (counter-clockwise) from the down-rest position of the pendulum.

(c) Let g be the acceleration of gravity, and assume that frictional forces can be neglected.

(d) Assume that the mass of the pendulum arm can be neglected.

Now **do the following**

A Using Newton's laws, **derive the equations for the pendulum.**

Hint: There are two forces acting on the mass M:

- The force of gravity (of magnitude Mg, pointing downwards).
- A force (of magnitude $F = F(t)$) acting along the arm of the pendulum.

The force F is not known a-priori, but it must have the exact magnitude to keep the distance from the mass to the pendulum hinge at the length $L = L(t)$. This is enough to determine this force.

B Consider the following situation: you have a mass tied up at the end of a string. The string goes through a small hole somewhere — say, the hole at the end of a fishing rod. Now, pull steadily on the string, shortening the string length from the hole to the mass (do not move the hole while this happens). You should observe that, quite often, you end up with the mass going around the "fishing rod", wrapping the string there. **Explain this behavior using the equations derived in A.** *(Note that real life is neither 2-D, nor frictionless: the equations tend to over-predict what happens).*

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C Study the stability of the $\theta = 0$ equilibrium position for the pendulum. Linearize the equations near this solution, and obtain an equation of the form

$$\frac{d^2\varphi}{dt^2} + V(t)\varphi = 0\,, \tag{1}$$

where $\varphi = L\theta$ and $V = V(t)$ is some "potential" obtained from L and its derivatives.

D Argue that, if L is sinusoidal, with small amplitude variations, then one can take

$$V = \Omega^2(1 + \epsilon\cos(\omega t))\,, \tag{2}$$

in (1), where ϵ is small. Then (1) becomes Mathieu's equation.

E Take $\Omega = 1$ in Mathieu's equation and use Floquet theory to study the stability of the pendulum. That is, calculate (numerically) the trace of the Floquet matrix as a function of ϵ and ω (say, for $0 \le \epsilon \le 0.3$ and $0.5 \le \omega \le 5$). Note that the period to use in the calculation is $2\pi/\omega$ — i.e.: the period of $V = V(t)$ — and that instability corresponds to $\alpha = \mathsf{trace}/2$ having magnitude bigger than one.

Alternatively: you can take $\omega = 1$, and then vary Ω and ϵ.

THE END.