18.385j/2.036j Problem List.

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Abstract

This is a list of problems for the MIT course 18.385j/2.036j. These problems may (or may not) be assigned as part of the course problem sets or exams.

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1 PDE Blow Up.

1.1 Statement: PDE Blow Up.

The purpose of this problem is to investigate an example where smooth solutions to a PDE cease to exist after finite time. In the lectures (see the remark below) we considered the problem:

$$u_t + u u_x = 0\,, \quad \text{with initial data} \quad u(x,0) = F(x)\,, \tag{1.1}$$

where
- $u = u(x,t)$ is a function of x (space) and t (time).
- u_t and u_x denote the partial derivatives of u with respect to t and x.

We saw that: **The solution to problem (1.1) ceases to exist at a finite time (the derivatives of u become infinite and, beyond that, u becomes multiple valued), whenever dF/dx is negative somewhere.** (1.2)

Consider now the problem:

$$u_t + u u_x = -u\,, \quad \text{with initial data} \quad u(x,0) = F(x)\,. \tag{1.3}$$

Show that the solution to this second problem ceases to exist at a finite time, provided that $dF/dx < C < 0$, where C is a finite (non-zero) constant. Again, what happens is that the derivatives become infinite. Calculate C.

Hint: *To show the result (quoted above) in class, we used* **two** *(related) approaches, both involving the characteristic curves. In the first we obtained the exact solution to the problem (in implicit form) and from that we obtained the result. The second approach went straight to the point and showed that u_x had to "blow up" on some characteristic curve. Both techniques will work here too, but the second approach is much simpler (for the purposes of what you are being asked to show).*

Remark: *In order to show the result in (1.2), we proceeded in two ways.*

(1-st). *Introduce the characteristic curves in space-time, which — for equation (1.1), are defined by $\frac{dx}{dt} = u(x,t)$. Equation (1.1) then yields $\frac{du}{dt} = 0$ along these curves, so we can write*

$$\left.\begin{array}{rclcrcl} \frac{dx}{dt} & = & u & \textit{and} & x(0) & = & \zeta \\ \frac{du}{dt} & = & 0 & \textit{and} & u(0) & = & F(\zeta) \end{array}\right\} \Longrightarrow \left\{\begin{array}{rcl} x & = & \zeta + F(\zeta)t \\ u & = & F(\zeta) \end{array}\right\} \tag{1.4}$$

for each characteristic, which we label by the value x takes for $t = 0$ — namely: ζ. Thus, for this problem, the characteristics are straight lines in space-time, and along each of them u is constant. Clearly, if the initial data F is decreasing anywhere,[1] then there will be characteristics that cross somewhere in space-time, for a finite time $t > 0$. But then the solution there would be multiple valued! The earliest time at which this happen would correspond to two infinitesimally distant (i.e. ζ and $\zeta + d\zeta$) characteristics crossing, at which point the drivative u_x would become $-\infty$.

Equation (1.4) has the following interpretation — in terms of what the shape of the solution (i.e. $y = u(x,t)$, for each fixed time) does as a function of time. Each value of u propagates at its own speed, that happens to be u. Thus, in places where u is decreasing, the shape steepens (values behind catch-up to the values ahead) and, eventually, an infinite slope arises at some point.

[1] Namely: $\frac{dF}{dx} < 0$ someplace.

(2-nd). *Introduce the characteristic curves, but now directly write an equation for how u_x changes along each characteristic. By taking the x partial derivative of equation (1.1), it is easy to see that*

$$\frac{dw}{dt} + w^2 = 0, \quad \textit{where} \quad w = u_x. \tag{1.5}$$

Along any characteristic where w starts negative, it reaches $-\infty$ in a finite time. Hence (1.2) follows.

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2 Critical Point Problems.

2.1 Statement: Saddles and Conservative Systems.

Saddles for conservative systems are very special, since the rate of expansion and contraction along the two principal directions are equal. The purpose of this problem is to guide you in the process of showing this.

Consider a phase plane autonomous system

$$\frac{dx}{dt} = f(x, y) \quad \text{and} \quad \frac{dy}{dt} = g(x, y), \tag{2.1}$$

where both f and g have continuous partial derivatives. We will now assume:

A. The origin $x = y = 0$ is an **isolated critical point;** in fact: a **saddle,** with eigenvalues for the linearized problem $\boxed{\lambda_1 > 0 > \lambda_2.}$

B. The system is **conservative** near the origin. Namely: there is a function $E = E(x, 0$ — defined in a neighborhood of the origin — which is a constant on the orbits. Furthermore, **E is continuously differentiable and $\nabla E \neq 0$, except at the origin.**

C. The origin is a **true saddle for E.** This means that E is twice differentiable at the origin, with $\boxed{\det(D^2E) < 0.}$

Show that: $\boxed{\lambda_1 = -\lambda_2.}$ **Equivalently, that:** $\boxed{f_x(0,0) + g_y(0,0) = 0.}$

Note (what does C mean?): *Since E is twice differentiable at the origin, we can write*[2]

$$E = E_0 + \frac{1}{2}a\,x^2 + b\,x\,y + \frac{1}{2}c\,y^2 + o\left((x^2+y^2)^{1.5}\right), \tag{2.2}$$

[2]This is, in fact, the definition of twice differentiable.

where E_0*,* a*,* b*, and* c *are constants. Then* $det(D^2E) < 0 \iff ac - b^2 < 0$ *— which means that the quadratic form* $Q = Q(x,y) = \frac{1}{2}a\,x^2 + b\,x\,y + \frac{1}{2}c\,y^2$ *above in (2.2) is non-degenerate and non-definite. Thus:* **(I)** *The level lines for* Q *govern the local behavior of the level lines for* E*, and* **(II)** Q *has a saddle at the origin, so its level lines yield a saddle. Therefore the surface* $z = E(x,y)$ *has a saddle at the origin, which is dominated by the quadratic terms in its Taylor expansion.*

As an example where this is not true, consider the system $\dot{x} = x$ *and* $\dot{y} = -2y$*, which has a saddle at the origin and the conserved quantity* $E = y\,x^2$*. But the surface* $z = E(x\,,y)$ does not have *a saddle at the origin. Another example is provided by the system* $\dot{x} = x$ *and* $\dot{y} = -3y$*, also with a saddle at the origin and a conserved quantity* $E = y\,x^3$*. In this case the surface* $z = E(x\,,y)$ *does have a saddle at the origin, but it is a very degenerate saddle.*[3]

Note (about ∇E): *From equation (2.2) is follows that*

$$\nabla E = \begin{pmatrix} a\,x + b\,y \\ b\,x + c\,y \end{pmatrix} + o\,(x^2 + y^2) = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + o\,(x^2 + y^2). \tag{2.3}$$

Thus, because the matrix of coefficients is non-degenerate, $\nabla E \neq 0$ *near (but not at) the origin.*

Hint: *Because* E *is conserved:*

$$Z = Z(x,y) = f\,E_x + g\,E_y \equiv 0. \quad \text{Why this?} \tag{2.4}$$

On the other hand, because f *and* g *are differentiable (and the origin is a critical point), we can write (for some constants* α*,* β*,* γ*, and* δ*):*

$$f = \alpha\,x + \beta\,y + o\,(x^2 + y^2) \quad \textit{and} \quad g = \gamma\,x + \delta\,y + o\,(x^2 + y^2). \tag{2.5}$$

Substitute these expansions — and (2.3), into (2.4) and then conclude that it must be $\alpha + \delta = 0$ *— which is precisely what you are being asked to show.*

Remark (general saddles): *Modulo a nonsingular linear transformation of the dependent variables* x *and* y*, and a change in the time scale (possibly including a time reversal), the equations for a linear system with a saddle can always be written in the form*

$$\frac{dx}{dt} = -x \quad \textit{and} \quad \frac{dy}{dt} = \nu y, \quad \textit{where } \nu \geq 1. \tag{2.6}$$

[3] You would not want to sit on a horse saddle made following this design!

Conserved quantities for this last system must have the form $\boxed{E = f(|x|^\nu\, y)}$ — *where* $f = f(\zeta)$ *is some arbitrary function. But then*

$$E_x = sign(x)\, x^{(\nu-1)}\, y\, f'(|x|^\nu\, y) \quad and \quad E_y = x^\nu\, f'(|x|^\nu\, y), \tag{2.7}$$

so that ∇E vanishes identically for $x = 0$, unless $\nu = 1$ (which is when the trace of the matrix of coefficients vanishes).

2.2 Statement: Critical Points for Conservative Systems.

This problem generalizes the result in the problem "Saddles and Conservative Systems" to arbitrary critical points for conservative systems in any dimension. For such points the the linear eigenvalues must add up to zero.

Consider an N-dimensional autonomous system

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \tag{2.8}$$

where $\mathbf{x}$ and $\mathbf{F}$ are N-vector valued, and $\mathbf{F}$ is continuously differentiable. **Assume** now that

A. The origin $\mathbf{x} = 0$ is a **non-degenerate critical point.** Namely

$$\mathbf{F} = \mathbf{A}\cdot\mathbf{x} + O(\|\mathbf{x}\|^2), \qquad \text{where } \mathbf{A} \text{ is a nonsingular } N\times N \text{ matrix.} \tag{2.9}$$

B. The system is **conservative** near the origin. Namely: there is a function $E = E(\mathbf{x})$ — defined in a neighborhood of the origin — which is a constant on the orbits.

C. E is continuously differentiable. Also twice differentiable at the origin, with:

$$E = \frac{1}{2}\,\mathbf{x}\cdot\mathbf{Q}\cdot\mathbf{x} + O(\|\mathbf{x}\|^3), \qquad \text{where } \mathbf{Q} \text{ is a symmetric, nonsingular, } N\times N \text{ matrix.} \tag{2.10}$$

Then show that: $\boxed{\mathbf{Tr}(\mathbf{A}) = 0.}$

Thus, the critical point can neither be an attractor nor a repeller, since in these cases all the eigenvalues have real parts with the same sign. In 2-D this only allows for either saddles — with the two eigenvalues of the same size — or centers. In 3-D, again, only two possibilities are allowed:

1. Saddle: real eigenvalues, one of opposite sign to the other two, all three adding up to zero.

2. Saddle-spiral: one real eigenvalue and two complex conjugate ones, with the real part of the complex conjugate eigenvalues equal to 1/2 the negative of the real eigenvalue.

Note (about ∇E): *From equation (2.10) is follows that*

$$\nabla E = \mathbf{Q} \cdot \mathbf{x} + O(\|\mathbf{x}\|^2). \tag{2.11}$$

Thus, because the matrix of coefficients is non-degenerate, $\nabla E \neq 0$ near (but not at) the origin.

Hint: *Because E is conserved:*

$$Z = Z(\mathbf{x}) = \mathbf{F} \cdot \nabla E \equiv 0. \quad \text{Why this?} \tag{2.12}$$

Substitute here the expansions in (2.9) and (2.11), and then conclude that it must be $\mathrm{Tr}(\mathbf{A}) = 0$.

IMPORTANT: *Let $\mathbf{M}$ be an $N \times N$ symmetric matrix, such that $\mathbf{x} \cdot \mathbf{M} \cdot \mathbf{x} = 0$ for every choice of $\mathbf{x}$. Then $\mathbf{M} = 0$. Note though: the fact that $\mathbf{M}$ is symmetric is* **crucial** *for this result.*

2.3 Statement: Structural Stability of Stars.

Consider a system in the plane

$$\left.\begin{aligned} \frac{dx}{dt} &= f(x, y)\,, \\ \frac{dy}{dt} &= g(x, y)\,, \end{aligned}\right\} \tag{2.13}$$

such that the origin $\mathbf{P} = (x, y) = (0, 0)$ is an isolated critical point, with the linearized system there having a stable star. Now consider the following two alternatives for the complete behavior of the system:

a) Linearized: **stable star**. Fully nonlinear: **stable spiral**.

b) Linearized: **stable star**. Fully nonlinear: **stable proper node**.

Which ones are possible? For each one that is possible, give an example of a system with the desired behavior. Otherwise, explain why you think the particular alternative cannot happen. In this case, how close can you get (produce an example that "almost" does it)?

Optional: Give thought to the nature of the perturbation you need: smooth[4] perturbations will not do the job, why? It turns out that the perturbations needed cannot even have a second derivative at the origin.[5] Can you give some argument in the direction of what is the "minimum" amount of singularity needed for the job? Note that arbitrarily singular perturbations cannot be allowed, since then the notion of a linearized problem will not make sense.

Recall the definitions:

1. For a linear system, a stable star is a point with a double eigenvalue of equal algebraic and geometric multiplicities. Thus its associated matrix is a multiple of the identity.

2. We say that a critical point for a nonlinear system is a node (spiral, whatever) if the **phase portrait NEAR the critical point can be "deformed" by a continuous transformation into the phase portrait for the corresponding linear system.** That is: the two phase portraits "look" qualitatively the same. For the purposes of this problem use this second "definition" (i.e.: do not worry about continuous transformations, just show that the key properties are the same). Thus, for example, **the origin is:**

2a. A (stable) spiral point if the orbits near the origin satisfy: $r \to 0$ and $\theta \to \infty$ (or $\theta \to -\infty$) as $t \to \infty$.

2b. A (stable) proper node if all the orbits near the origin approach it as $t \to \infty$, and there are two special directions ($\theta = \pm\theta_1$ and $\theta = \pm\theta_2$) such that:

- There is exactly one orbit such that $\theta \to +\theta_1$ as $t \to \infty$.
- There is exactly one orbit such that $\theta \to -\theta_1$ as $t \to \infty$.
- For all other orbits: As $t \to \infty$, either $\theta \to +\theta_2$, or $\theta \to -\theta_2$

For the unstable cases, simply change $t \to \infty$ to $t \to \infty$ in the definitions above.

Hint 1 *Consider first small* linear *perturbations to a linear systems that can cause the appropriate changes. Then write a system where perturbations of the same form are introduced by a nonlinearity. The nonlinearity will have to be* **small**, *so that it vanishes faster than the linear terms as the origin is approached; but do not make it vanish too fast, else it will not do the job! In fact, you should find*

[4]Smooth means that the perturbation has infinitely many derivatives.

[5]One derivative is needed to have the linearization make sense.

that it must vanish so "slowly", that the resulting function has second derivatives that "blow up" at the origin.

3 Bifurcation Problems.

3.1 Statement: Bifurcations of a Critical Point for a 1D map.

When studying bifurcations of critical points in the lectures, we argued that we could understand most of the relevant possibilities simply by looking at 1-D flows. The argument was based on the idea that, generally, a stable critical point will become unstable in only one direction at a bifurcation — so that the flow will be trivial in all the other directions, and we need to concentrate only on what occurs in the unstable direction. The only important situation that is missed by this argument, is the case when two complex conjugate eigenvalues become unstable (Hopf bifurcation.) Because in real valued systems eigenvalues arise either in complex conjugate pairs or as single real ones, the cases where only one (or a complex pair) go unstable are the the only ones likely to occur (barring situations with special symmetries that "lock" eigenvalues into synchronous behavior.)

The same argument can be made when studying bifurcations of limit cycles in any number of dimensions. In this case one considers the Poincaré map near the limit cycle,[6] with the role of the eigenvalues taken over by the Floquet multipliers. Again, we can argue that we can understand a good deal of what happens by replacing the (multi-dimensional) Poincaré map by a one dimensional map with a stable fixed point, and then asking what can happen if the fixed point changes stability as some parameter is varied. In this problem you will be asked to do this.

Remark 1 *Again, some important cases are missed by this approach. Namely: the cases where a pair of complex Floquet multipliers becomes unstable (a Hopf bifurcation of a limit cycle) and the cases where a bifurcation occurs because of an interaction of the limit cycle with some other object (say, a critical point.) Several examples of these situations can be found in the book, in section 8.4 (Global Bifurcations of Cycles.)*

Consider a one dimensional (smooth) map from the real line to itself

$$x \longrightarrow y = f(x, \mu)\,, \qquad (3.1)$$

[6] The limit cycle is a fixed point for this map.

that depends on some (real valued) parameter μ. **Assume that $x = 0$ is a fixed point for all values of μ (that is, $f(0, \mu) \equiv 0$).** Furthermore, assume that $x = 0$ is stable for $\mu < 0$ and unstable for $\mu > 0$. That is:

$$|\frac{\partial f}{\partial x}(0, \mu)| \quad < \quad 1 \quad \text{for } \mu < 0 \,, \quad \text{and} \tag{3.2}$$

$$|\frac{\partial f}{\partial x}(0, \mu)| \quad > \quad 1 \quad \text{for } \mu > 0 \,. \tag{3.3}$$

A further assumption, that involves no loss of generality (since the parameter μ can always be re-defined so it is true) is that

$$\frac{\partial^2 f}{\partial x \partial \mu}(0, 0) \neq 0 \,. \tag{3.4}$$

This guarantees that the loss of stability is linear in μ, as μ crosses zero. This is what is called a *transversality condition.* It means this: If you graph of the Floquet multiplier $\frac{\partial f}{\partial x}(0, \mu)$ as a function of μ, then the resulting curve crosses the unit circle transversaly (i.e.: it is not tangent to it) for $\mu = 0$.

By doing an appropriate expansion of the map f for x and μ small (or by any other means), show that (generally[7]) the following happens:

a) For $\frac{\partial f}{\partial x}(0, 0) = 1$, either:

a1) *Transcritical bifurcation (no special symmetries assumed for f):* There exists another fixed point, $x_* = x_*(\mu) = O(\mu)$, such that: $x_* > 0$ is unstable for $\mu < 0$ and $x_* < 0$ is stable for $\mu > 0$. The two points "collide" at $\mu = 0$ and exchange stability.

a2) *Supercritical or soft bifurcation, f is an odd function of x:* Two stable fixed points exist for $\mu > 0$, one on each side of $x = 0$, at a distance $O(\sqrt{\mu})$. All three points merge for $\mu = 0$.

a3) *Subcritical or hard bifurcation, f is an odd function of x:* Two unstable fixed points exist for $\mu < 0$, one on each side of $x = 0$, at a distance $O(\sqrt{-\mu})$. All three points merge for $\mu = 0$.

Explain what all of this means in the context of a limit cycle.

[7]There are special conditions under which all this fails. You will have to find them as part of your analysis. What are they?

b) **For** $\dfrac{\partial f}{\partial x}(0,0) = -1$ **(no special symmetries assumed for** f**), either:**

b1) *Supercritical or soft flip bifurcation:* For $\mu > 0$ two points exist $x_1(\mu) \approx -x_2(\mu)$ (both of them of size $O(\sqrt{\mu})$), on each side of the fixed point $x = 0$, such that $x_2 = f(x_1, \mu)$ and $x_1 = f(x_2, \mu)$. Thus $\{x_1,\, x_2\}$ is a period two orbit for the map (3.1). **Show that this orbit is stable.**

b2) *Subcritical or hard flip bifurcation:* For $\mu < 0$ two points exist $x_1(\mu) \approx -x_2(\mu)$ (both of them of size $O(\sqrt{-\mu})$), on each side of the fixed point $x = 0$, such that $x_2 = f(x_1, \mu)$ and $x_1 = f(x_2, \mu)$. Thus $\{x_1,\, x_2\}$ is a period two orbit for the map (3.1). **Show that this orbit is unstable.**

In the context of a limit cycle, these are the period doubling bifurcations. Explain.

Hint 1 *If you expand f in a Taylor expansion near $x = 0$ and $\mu = 0$, up to the leading order beyond the trivial first term ($f \sim \pm x$), and you are careful about keeping* **all** *the relevant terms (and nothing else), and you are careful about identifying certain terms that are known to vanish, the problem reduces to (mostly) trivial high-school algebra. In figuring out what to keep and what not to keep, note that you have two small quantities (x and μ), whose sizes are related. The whole thing is very much like the Hopf bifurcation expansion calculation done in the lectures (or in the course notes), but much, much simpler computationally.*

In part (b) you will have to keep in the expansion not just the leading order terms beyond the trivial first term, but the next order as well. Notice that, in this case, you will be trying to find solutions of $f(f(x, \mu), \mu) = x$. When you calculate $f(f(x, \mu), \mu)$, you will see that the second order terms cancel out — thus the need to keep one extra term in the expansion. This is, exactly, the same phenomena that forced the need to carry the Hopf bifurcation calculation (in the lectures, etc.) to third order.

Just to make the notation simple, use the letters a, b, … etc., to denote the various partial derivatives $\dfrac{\partial^{n+m} f}{\partial^n x \partial^m \mu}(0,\, 0)$ that you will need in the Taylor expansion. Please, do not use the notation $\dfrac{\partial^{n+m} f}{\partial^n x \partial^m \mu}(0,\, 0)$ everywhere; just once when you define a, b, …, etc.

4 Approximation Techniques Problems.

4.1 Statement: Multiple scales and limit cycle problem 01.

Consider the equation

$$\frac{d^2x}{dt^2} - \epsilon\cos(x)\frac{dx}{dt} + \frac{1}{\sqrt{\epsilon}}\sin(\sqrt{\epsilon}x) = 0\,, \quad \text{where} \quad \boxed{0 < \epsilon \ll 1.} \tag{4.1}$$

Use a multiple scales analysis to calculate the frequency, stability and amplitude of the limit cycle (the frequency up to the first correction beyond linear and the amplitude up to leading order).

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5 Poincaré Map Problems.

5.1 Statement: Simple Poincaré Map for a limit cycle.

Consider the following autonomous phase plane system

$$\left.\begin{aligned} \frac{dx}{dt} &= (x^2+y^4)\left(x - \tfrac{1}{4}x^3 - x^2\,y - x\,y^2 - 4\,y^3\right), \\ \frac{dy}{dt} &= (x^2+y^4)\left(y + \tfrac{1}{4}x^3 - \tfrac{1}{4}x^2\,y + x\,y^2 - y^3\right). \end{aligned}\right\} \tag{5.1}$$

This system has a periodic solution, which can be written in the form

$$x = 2\cos\Phi, \quad y = \sin\Phi, \quad \text{where} \quad \frac{d\Phi}{dt} = 2\,(x^2+y^4) = 2\,(1+\cos^2\Phi)^2. \tag{5.2}$$

This solution corresponds to the orbit going through the point $x = 0$, $y = 1$ in the phase plane — which orbit is an ellipse, as (5.2) shows.[8]

Construct (either analytically or numerically) a Poincaré map near this orbit, and use it to show that the orbit is a stable limit cycle.

Define the Poincaré map $\boxed{z \longrightarrow u = P(z)}$ as follows:

- For every sufficiently small z, let $x = X(t, z)$ and $y = Y(t, z)$ be the solution of the system in (5.1) defined by $X(0, z) = 0$ and $Y(0, z) = 1 + z$.

[8]Notice that Φ is a strictly increasing function of time.

• For this solution the polar angle θ in the phase plane is an increasing function of time, starting at $\theta = (1/2)\,\pi$ for $t = 0$. Thus, there will be a time $t = t_z$ at which the solution will reach $\theta = (5/2)\,\pi$ — notice that t_z is a function of z .... Then take $\boxed{u = Y(t_z, z) - 1.}$

Hint: *Because t_z is a function of z, unknown a priori, the definition of the Poincaré map above is a bit awkward to implement. To avoid the issue of having to calculate t_z for each solution, it is a good idea to use a parameter other than time to describe the orbits. For example, if the equations are written in terms of a parameter such as the polar angle — namely $dx/d\theta = F(x, y)$ and $dy/d\theta = G(x, y)$, then the Poincaré map is easier to describe, as θ varies from $\theta = (1/2)\pi$ to $\theta = (5/2)\pi$ in every one of the orbits needed to compute $u = P(z)$.*

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6 Fourier Series Problems.

6.1 Statement: Fourier Series.

This problem objective is to "experimentally" **study how Fourier Series converge.** For this purpose you should use the following MatLab scripts (which you can download from the WEB page):

readmeFouSer.m

fourierSC.m

FSFun.m

FSoption.m

FSoptionP.m

heatSln.m

Put them in a directory and start MatLab in that directory. The help command will work then for these scripts. For example:

help readmeFouSer

will give you a short description of all the scripts. Each script has it's own more detailed description. The *script you need is fourierSC.* The others (except for heatSln) are helper scripts.

Note: *you can alter FSFun.m and write there any function for which you want to investigate the Fourier Series (this will allow you to go beyond the preselected options).*

Notice also that fourierSC makes tons of plots (they will show up one on top of the other, so you'll need to "uncover" them).

This is what you should do:

Use the script fourierSC and report which sort of "patterns" do you see in the way Fourier Series converge. Experiment with the various choices. Look at the plots you'll get and think: what is happening? Note that many plots useful in figuring out how fast things converge (i.e. how fast do the Fourier coefficients vanish as $n \to \infty$) will be made by the scripts.

Look at the plots, look for patterns and trends. Make an hypothesis as to what is happening and then check it by further experimentation (use for this the script FSFun to produce functions where you can test your hypothesis). Then write your conclusions in the answers. Describe the evidence for your conclusions too; no proofs required, numerical evidence is enough. Think of it in the same way that you would think in the situation of an experimenter in a lab trying to figure out what happens in some problem.

By the way: you can use a few plots in your answer if you want, but do not go wild on this.

Note: Anything smaller than about 10^{-14} is numerical error, ignore it!

You will see graphs of the **power spectrum** appearing. This is just

$$\sqrt{c_n^2 + s_n^2},$$

where c_n and s_n are the coefficients of $\cos(nx)$ and $\sin(nx)$ in the Fourier Series. It gives you an idea of "how important" the n-th mode is in the Fourier expansion. The name follows from the fact that in many physical situations you can interpret the square of the amplitude of the n-th Fourier coefficient as the amount of energy in the n-th mode of the solution (this is the case for the wave equation, for example).

7 Hamiltonian Systems Problems.

7.1 Statement: Mass in 2D with a central force.

This problem is a generalization of problem 8.6.7 (page 295) in the book by Strogatz.

Consider the motion of a particle of mass m in the plane, subject to a central force given by a potential $V = V(r)$ — where $r = \sqrt{x^2 + y^2}$. The equations are then:

$$m\frac{d^2x}{dt^2} = -\frac{\partial}{\partial x}V \quad \text{and} \quad m\frac{d^2y}{dt^2} = -\frac{\partial}{\partial y}V \,. \tag{7.1}$$

We will **assume** that:

- $V'(r) = \frac{d}{dr}V(r) > 0$ (force always points towards center.)
- $V(r) \to \infty$ as $r \to \infty$.
- $r^2 V(r) \to 0$ as $r \to 0$.

a) Show that the equations can be written (using polar coordinates) in the form:

$$m\frac{d^2r}{dt^2} = \frac{h^2}{mr^3} - V'(r) \qquad \text{and} \qquad \frac{d\theta}{dt} = \frac{h}{mr^2}\,, \tag{7.2}$$

where h is a constant (the angular momentum.)

b) Show that the system has solutions of the form $r = r_0$ and $\frac{d\theta}{dt} = \omega_0$, corresponding to uniform circular motion of radius r_0 and frequency ω_0. Find formulas for r_0 and ω_0 for the case when $h > 0$ and $V = kr$, where $k > 0$ is a constant.

c) Find the frequency ω_1 for small radial oscillations about a circular orbit and show that (generally) these small radial oscillations give rise to quasiperiodic motion. That is, the frequencies ω_0 and ω_1 are not rationally related. In particular, calculate ω_1/ω_0 for the case $V = kr$.

d) Generalize **(c)** to the case of finite amplitudes. In fact, show that the general solution of (7.1) — excluding the case when $h = 0$ — is either periodic or quasiperiodic with two periods (with the second case being the generic one). In particular, this means that the motion in this system is never chaotic (this is an example of a **Completely Integrable Hamiltonian System**.) *Hint: analyze first the equation for the radial motion and show that the solutions are periodic. Then investigate the angular variable and notice that the orbital period (the time it takes θ to change by 2π) will, generally, not be rationally related to the period in the radial variable.*

8 Floquet Theory Problems.

8.1 Statement: Balancing a broom.

Consider the problem of balancing a broom upright, by placing it on a surface that moves up and down in some prescribed manner. Specifically:

Assume a rough flat horizontal surface, which oscillates up and down following some prescribed law (that is, at any time the surface can be described by the equation $y = Y(t)$, where y is the vertical coordinate, and Y is some oscillatory function). On this surface we place a broom, in upright position, with the sweeping side pointing up.[9] **Question: Can we prescribe Y in such a way that the broom remains upright — i.e.: the position is stable?**

In order to answer the question, consider the following idealized situation:

- **A)** Replace the broom by a mass m, placed at the upper end of a (massless) rigid rod of length L. Let the displacement of the rod from the vertical position be given by the angle θ, with $\theta = 0$ corresponding to the rod standing vertical, and the mass on the upper end.
- **B)** The bottom of the rod is attached to a hinge that allows it to rotate in a plane. Thus the motion of the rod is restricted to occur on a plane.
- **C)** Assume that friction can be neglected.
- **D)** The hinge to which the rod is attached oscillates up and down, with position $x = 0$ and $y = Y(t)$ — x is the horizontal coordinate on the plane where the rod moves. The mass is then at $x = L\,\sin(\theta)$ and $y = Y + L\,\cos(\theta)$ — we measure angles clockwise from the top.

Now, do the following:

(1)

Use Newton's laws to derive the equation of motion for the mass m. You should obtain a second order ODE for the angle θ, with coefficients depending on the parameters g (the acceleration of gravity) and the length of the rod L — in addition to the forcing function $Y = Y(t)$.

[9] Because the surface is rough, the contact point of the broom with the surface will not move relative to the surface.

Hint: Only two forces act on the mass m, namely: gravity and a force $F = F(t)$ along the rod. The force F has just the right strength to keep the (rigid) rod at constant length L — this is enough to determine F, though you do not need to calculate it.

(2)

You should notice that adding a constant velocity to the hinge motion (that is: $Y \to Y + v\,t$, where v is a constant) does not change the equation of motion. Why should this be so? What physical principle is involved?

(3)

Write down the (linearized) equations for small perturbations of the equilibrium position ($\theta = 0$) that we wish stabilized. **Stability occurs if and only if $Y = Y(t)$ can be selected so that the solutions of this linear equation do not grow in time** — strictly speaking we should also consider the possible effects of nonlinearity, but we will ignore this issue here.

(4)

You should notice that it is possible to stabilize $\theta = 0$ by taking $Y = -\frac{1}{2}\,a\,t^2$, where $a > 0$ is a constant acceleration. How large does a have to be for this to happen? Give a justification of this result based on physical reasoning, without involving any equations (this is something you should have been able to predict before you wrote a single equation).

(5)

Of course, the "solution" found in **(4)** is not very satisfactory, since Y grows without bound in it. Consider now oscillatory forcing functions of the form:

$$Y = \ell\,\cos(\omega\,t)\,, \tag{8.1}$$

where $\ell > 0$ and $\omega > 0$ are constants (with dimensions of length and time^{-1}, respectively).

The objective is to find conditions on (ℓ, ω) that guarantee stability. (8.2)

The next steps will lead you through this process, but first: **Nondimensionalize the (linearized) stability equation.** In doing so it is convenient to use the time scale provided by the forcing to nondimensionalize time — i.e.: let the nondimensional time be $\boxed{\tau = \omega\, t\,.}$

This step should lead you to an equation describing the evolution of the angle θ (valid for small angles), involving two nondimensional parameters. One of them, $\boxed{\epsilon = \ell/L,}$ measures the amplitude of the oscillations in terms of the length of the rod. The other measures the time scale of the forcing (as given by $1/\omega$) in terms of the time scale of the gravitational instability — a function of g and L. **Call this second parameter** $\boxed{\mu}$ — note that in the equation only μ^2 appears, not μ itself.

(6)

Find the stability range for μ as a function of ϵ, for the values $0 < \epsilon \leq 0.6$ — it is enough to pick a few values of ϵ, say $\epsilon = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5,\ 0.6$, and then to compute the stability range for each of them.

Note/hint: This step will require not just analysis, but some numerical computation. So as not to be forced to explore all possible values of μ when looking for the stability ranges (numerically an impossible task), you should notice that the analysis for $\epsilon = 0$ can be done exactly — and should provide you with a good hint as to where to look.

(7)

Write the period $p = \dfrac{2\pi}{\omega}$ of the forcing, in terms of the nondimensional parameter μ, and the parameters g and L. The results of **part (6)** should provide you with the period ranges (for a given oscillation amplitude) where stability occurs. Use this information to provide a rough explanation of why it is relatively easy to balance a broom on the palm of your hand (using the strategy outlined in this problem — try it), and why you will not be able to balance a pencil.

(8)

For $0 \leq \epsilon \ll 1$ and $0 \leq \mu \ll 1$ you should be able to obtain analytical approximations for the stable ranges. Do so, and compare your results with those of **part (6)**.

Hint: Floquet theory provides a function (the Floquet Trace $\alpha = \alpha(\mu, \epsilon)$) that characterizes linearized stability — stability if and only if $|\alpha| \leq 1$. Compute this function for μ and ϵ small.

8.2 Statement: Variable Length Pendulum.

Consider a pendulum (in a plane), whose arm length $L > 0$ changes in time (i.e.: $L = L(t)$). To make matters more precise:

(a) Let the **hinge** for the pendulum be at origin in the plane: $x = y = 0$.

(b) Let the mass M for the pendulum be at $x = L\sin\theta$ and $y = -L\cos\theta$, where θ is the angle measured (counter-clockwise) from the down-rest position of the pendulum.

(c) Let g be the acceleration of gravity, and assume that frictional forces can be neglected.

(d) Assume that the mass of the pendulum arm can be neglected.

Now **do the following**

A Using Newton's laws, **derive the equations for the pendulum.**

Hint: There are two forces acting on the mass M:

- The force of gravity (of magnitude Mg, pointing downwards).
- A force (of magnitude $F = F(t)$) acting along the arm of the pendulum.

The force F is not known a-priori, but it must have the exact magnitude to keep the distance from the mass to the pendulum hinge at the length $L = L(t)$. This is enough to determine this force.

B Consider the following situation: you have a mass tied up at the end of a string. The string goes through a small hole somewhere — say, the hole at the end of a fishing rod. Now, pull steadily on the string, shortening the string length from the hole to the mass (do not move the hole while this happens). You should observe that, quite often, you end up with the mass going around the "fishing rod", wrapping the string there. **Explain this behavior using the equations derived in A.** *(Note that real life is neither 2-D, nor frictionless: the equations tend to over-predict what happens).*

C Study the stability of the $\theta = 0$ equilibrium position for the pendulum. Linearize the equations near this solution, and obtain an equation of the form

$$\frac{d^2\varphi}{dt^2} + V(t)\varphi = 0\,, \tag{8.3}$$

where $\varphi = L\theta$ and $V = V(t)$ is some "potential" obtained from L and its derivatives.

D Argue that, if $L = L_0\,(1 + \delta\cos(\omega\, t))$ is sinusoidal,[10] with small amplitude $0 < \delta \ll 1$ variations , then one can take

$$V = \Omega^2(1 + \epsilon\cos(\omega t))\,, \tag{8.4}$$

in (8.3), where ϵ is small. Then (8.3) becomes Mathieu's equation. Non-dimensionalize the equation, so it takes the form

$$\frac{d^2\phi}{d\tau^2} + (\mu^2 + \delta\cos\tau)\,\phi = 0\,, \tag{8.5}$$

for some ϕ, μ and τ.

E Consider now Mathieu's equation and use Floquet theory to study the stability of the pendulum. That is, calculate (numerically) the trace of the Floquet matrix as a function of μ and δ. Note that the period (in terms of the variable τ) to use in the calculation is 2π, and that instability corresponds to $\alpha = \text{trace}/2$ having magnitude bigger than one. **HINT:** calculate (and plot) the trace as a function of μ in the interval $0 \leq \mu \leq 2$ for various values of δ, say $\delta = 0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5, \ldots$ — **give arguments explaining why one should expect this strategy to work "a priori", i.e. before you carry it through.**

9 Fractals and Dimension Problems.

9.1 Statement: Generalized Cantor Sets.

Suppose that we construct a new kind of Cantor set, by removing the middle **half** of each subinterval, rather than the middle third.

a) Show that the "length" of the resulting set still vanishes, as in the case of the regular Cantor set.

b) Find the similarity dimension of the set.

[10] Here $L_0 > 0$, $\omega > 0$, and $\delta > 0$ are constants.

c) Generalize the construction so as to produce a Cantor set with zero length and with a similarity dimension that can be picked as any arbitrary number in $0 < d < 1$.

9.2 Statement: Decimal Shift Map.

This is, basically, the same as problems 10.3.7 and 10.3.8 (pp. 390-391) in the book by Strogatz.

Consider the **decimal shift map** on the unit interval, given by:

$$x_{n+1} = S(x_n) = 10\, x_n \pmod 1\,. \tag{9.1}$$

As usual, "mod 1" means that we look only at the non-integer part of the variable. Thus, for example, $S(0.263) = 0.63$ and $S(0.0417) = 0.417$.

a) Draw the graph of the map S.

b) Find **all** the fixed points. *Hint: write x_n in decimal form.*

c) Show that the map has periodic points of all periods, but that all of them are unstable.

Hints:

- *For the first part, give an explicit example of a period p point, for each integer $p > 1$.*
- *For the second part, show that given x_0 on a periodic orbit (i.e.: the orbit given by (9.1), starting at x_0, is periodic) then $\delta > 0$ can be found (where δ can be taken arbitrarily small) such that the orbit starting at $X_0 = x_0 + \delta$ — call it $\{X_n\}$ — satisfies*

$$\max_{n \geq 0} |X_n - x_n| \geq \frac{1}{2}\,.$$

In fact, you can choose δ so that $|X_n - x_n| \geq 0.4$ for all n large enough!

d) Show that the map has infinitely many aperiodic orbits. *Hint: think irrational numbers.*

e) Considering the rate of separation between two nearby orbits, show that the map has sensitive dependence on initial conditions.

f) An orbit $\{x_n\}$ is said to be *dense* if it eventually gets arbitrarily close to every point[11] in phase space — in this case: the unit interval. More precisely, the orbit $\{x_n\}$ is **dense** if given any $\epsilon > 0$ and any point $z \in [0, 1]$, there is some finite n such that $|x_n - z| < \epsilon$.

Explicitly construct a dense orbit for the decimal shift map.

9.3 Statement: Coastline Fractal.

In this problem we construct a fractal that is a *very idealized* caricature of what a coastline looks like. The construction proceeds by iteration of a basic process, which we describe next.

We start with a simple curve, Γ_0, and apply to it a simple process, that yields a new curve Γ_1. This new curve is made up of several parts, each of which is a scaled down copy of Γ_0. The same simple process is then applied to each of these parts, yielding Γ_2. Then we iterate, to obtain in this fashion a series of curves Γ_n, for $n = 0,\ 1,\ 2,\ 3\ldots$. The fractal is then the limit of this process: $\Gamma = \lim_{n\to\infty} \Gamma_n$ — provided the limit exists.

For the "coastline fractal" we **start by picking an angle** $0 < \theta < \pi$**, and a length** $R_0 > 0$**.** Then the first curve is:

$$\boxed{\Gamma_0 = \textbf{Circular arc of radius } R_0\textbf{, subtending an angle } \theta\textbf{.}} \tag{9.2}$$

Next **divide** Γ_0 **into three equal sub-arcs, each subtending an angle** $\theta/3$**, and replace each of these pieces by a properly scaled version of** Γ_0**. This yields** Γ_1**.** The process is then repeated on each of the three pieces making up Γ_1, so as to obtain Γ_2, and so on ad infinitum. The first two steps in this construction are illustrated in figure 1.

The issue of whether or not the limit $\lim_{n\to\infty} \Gamma_n$ exists is easy to settle. Consider an arbitrary radial line within the circle sector associated with Γ_0, and the intersection of this line with Γ_n. It should be clear that this intersection is unique. Let d_n be the distance of this intersection from the origin of the radial line. Then $\{d_n\}$ is an increasing, bounded sequence — so it has a limit. This limit defines a point along the radial line. The set of all these points is the fractal Γ.

[11] Such an orbit has to hop around rather crazily!

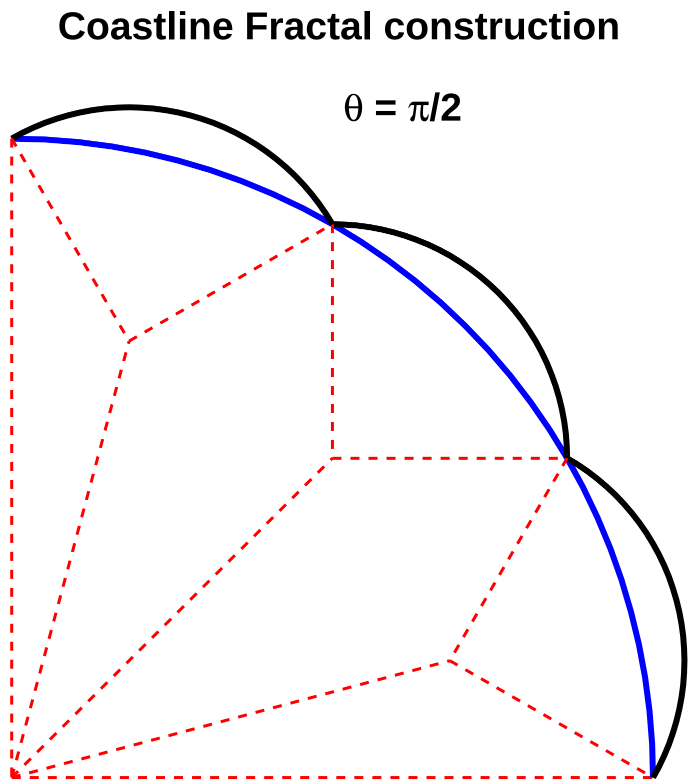

Figure 1: This figure illustrates the first two steps in the construction of the coastline fractal. That is, the curve Γ_0 (solid blue) and the curve Γ_1 (solid black). The dashed red lines indicate various radial lines useful in the construction.

Now do the following:

(1)

For each $n = 0,\, 1,\, 2,\, 3\ldots$, calculate the length ℓ_n of the curve Γ_n. What is the "length" of Γ?

(2)

Calculate the fractal dimension (self-similar or box) of Γ.

Hint:

The first thing you will need to calculate is the "scaling" factor between Γ_0 and each of the three parts that make up Γ_1. With this scaling factor $0 < S_c = S_c(\theta) < 1$, everything else follows.

Notes:

Real coastlines are not this simple, of course. At the very least the number of parts into which each sector is divided should not be a constant (3 here), nor should the parts be equal in size, nor should they all subtend the same angle θ. But further: the sectors need not be exactly circular — though, this is probably not a terrible approximation.

THE END.