MIT 18.385j/2.036j

First Problem Set

Suggested Readings (textbook): Chapters 1-2-3.
Suggested Problems (textbook):
Ch. 2: 2.2.9 2.2.12 2.2.13 2.3.3 2.4.9 2.6.1 2.8.3 2.8.5
Ch. 3: 3.3.1 3.4.5 3.4.7 3.4.8 3.4.9 3.4.10

Problems to hand in for grading (textbook):
Ch. 2: 2.2.8 2.2.10 2.3.2 2.5.4 2.5.5 2.5.6
Ch. 3: 3.2.6 3.2.7 3.3.2 3.4.6

PROBLEM TO HAND IN FOR GRADING (not in textbook):

PDE_Blow_Up

In the lectures we considered the PDE problem initial value problem:

$$u_t + u*u_x = 0; \quad u(x, 0) = F(x).$$

Notation:
1) u_t and u_x are the partial derivatives, with respect to t and x (resp.).
2) t is time and x is space.
3) * is the multiplication operator.
4) ^ denotes taking a power [u^2 is the square of u].
4) u = u(x, t) is a function of x and t.

We showed that the solution to this problem ceased to exist at a finite time (the derivatives of u become infinite and, beyond that, u becomes multiple valued) whenever dF/dx was negative anywhere.

This was shown "graphically". It can be shown analytically as follows:

--- A. Consider the CHARACTERISTIC CURVES dx/dt = u(x, t), as instroduced in the lecture.

--- B. Along each characteristic curve, one has du/dt = 0, as shown in class. Now, let v = u_x. Then v satisfies the equation [obtained by taking the partial derivative with respect to x of the equation for u]:

$$v_t + u*v_x + v^2 = 0.$$

Thus, along characteristics: dv/dt + v^2 = 0. Thus, if v is negative anywhere, v develops an infinity in finite time. But the initial conditions for v, along the characteristic such that x(0) = x_0, is v(0) = dF/dx(x_0).

Hence the conclusion follows: the solution u = u(x, t) to the problem ceases to exist at a finite time (with the derivative u_x of u becoming infinite somewhere) whenever dF/dx is negative anywhere.

CONSIDER NOW THE PROBLEM:

$$u_t + u*u_x = -u; \quad u(x, 0) = F(x).$$

Show that the solution to this second problem ceases to exist at a finite time, provided that dF/dx < C < 0, where C is a finite (non-zero) constant. Again, what happens is that the derivatives become infinite. Calculate C.

Hint: Use an approach analog to the one used above: get an ODE for the derivative v = u_x along the characteristics, and study the conditions under which the solutions of the ODE blow-up in a finite time.

A graphical approach for how the solution to u_t + u*u_x = -u behaves in time will also work, but the approach using the ODE for v along characteristics turns out to be simpler.
