• Coastline Fractal.

Statement:

In this problem we construct a fractal that is a *very idealized* caricature of what a coastline looks like. The construction proceeds by iteration of a basic process, which we describe next.

We start with a simple curve, Γ_0, and apply to it a simple process, that yields a new curve Γ_1. This new curve is made up of several parts, each of which is a scaled down copy of Γ_0. The same simple process is then applied to each of these parts, yielding Γ_2. Then we iterate, to obtain in this fashion a series of curves Γ_n, for $n = 0, 1, 2, 3 \ldots$. The fractal is then the limit of this process: $\Gamma = \lim_{n\to\infty} \Gamma_n$ — provided the limit exists.

For the "coastline fractal" we **start by picking an angle** $0 < \theta < \pi$**, and a length** $R_0 > 0$**.** Then the first curve is:

$$\boxed{\Gamma_0 = \text{Circular arc of radius } R_0\text{, subtending an angle } \theta.} \tag{1}$$

Next **divide** Γ_0 **into three equal sub-arcs, each subtending an angle** $\theta/3$**, and replace each of these pieces by a properly scaled version of** Γ_0**. This yields** Γ_1**.** The process is then repeated on each of the three pieces making up Γ_1, so as to obtain Γ_2, and so on ad infinitum. The first two steps in this construction are illustrated in figure 1.

The issue of whether or not the limit $\lim_{n\to\infty} \Gamma_n$ exists is easy to settle. Consider an arbitrary radial line within the circle sector associated with Γ_0, and the intersection of this line with Γ_n. It should be clear that this intersection is unique. Let d_n be the distance of this intersection from the origin of the radial line. Then $\{d_n\}$ is an increasing, bounded sequence — so it has a limit. This limit defines a point along the radial line. The set of all these points is the fractal Γ.

Now do the following:

(1)

For each $n = 0, 1, 2, 3 \ldots$, calculate the length ℓ_n of the curve Γ_n. What is the "length" of Γ?

(2)

Calculate the fractal dimension (self-similar or box) of Γ.

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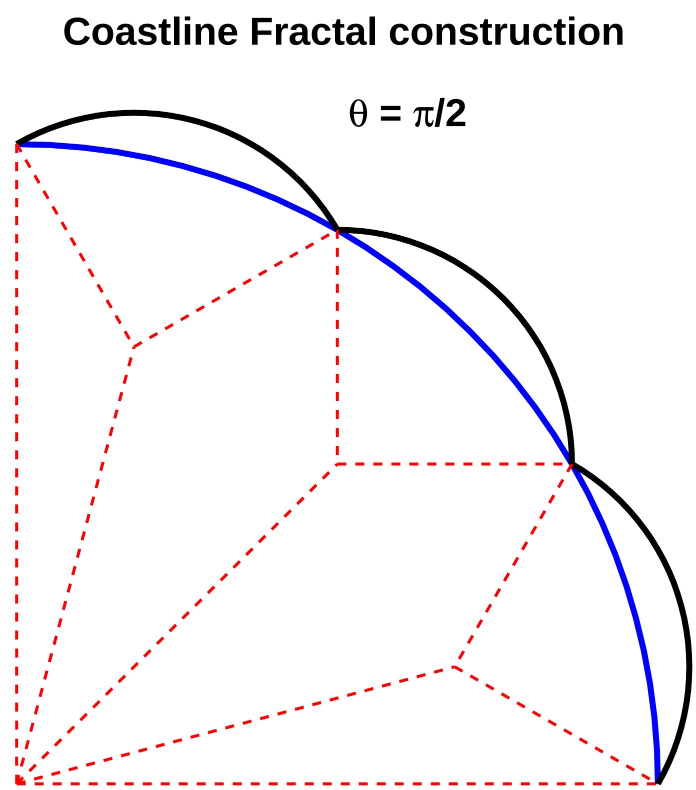

Figure 1: This figure illustrates the first two steps in the construction of the coastline fractal. That is, the curve Γ_0 (solid blue) and the curve Γ_1 (solid black). The dashed red lines indicate various radial lines useful in the construction.

Hint:

The first thing you will need to calculate is the “scaling” factor between Γ_0 and each of the three parts that make up Γ_1. With this scaling factor $0 < S_c = S_c(\theta) < 1$, everything else follows.

Notes:

Real coastlines are not this simple, of course. At the very least the number of parts into which each sector is divided should not be a constant (3 here), nor should the parts be equal in size, nor should they all subtend the same angle θ. But further: the sectors need not be exactly circular — though, this is probably not a terrible approximation.

THE END.