Problem Set Number 10, 18.385j/2.036j
MIT (Fall 2014)

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1 Problem 09.06.02 - Strogatz. Pecora and Carroll's approach

Statement for problem 09.06.02

Pecora and Carroll's approach for signal transmission/reception using the Lorenz system. In the pioneering work of Pecora and Carroll[1] one of the receiver variables is simply set *equal* to the corresponding transmitter variable. For instance, if $x(t)$ is used as the transmitter drive signal, then the receiver equations are

$$\left.\begin{aligned} x_r(t) &\equiv x(t), \\ \frac{dy_r}{dt} &= r\,x(t) - y_r - x(t)\,z_r, \\ \frac{dz_r}{dt} &= x(t)\,y_r - b\,z_r, \end{aligned}\right\} \tag{1.1}$$

where the first equation is **not** a differential equation.[2] Their numerical simulations, and a heuristic argument, suggested that $y_r(t) \to y(t)$ and $z_r(t) \to z(t)$ as $t \to \infty$, even if there were differences in the initial conditions.

Here are the steps for simple proof of the result stated above, due to He and Vaidya.[3]

A. Show that the error dynamics are governed by:

$$\left.\begin{aligned} e_x(t) &\equiv 0, \\ \frac{de_y}{dt} &= -e_y - x(t)\,e_z, \\ \frac{de_z}{dt} &= x(t)\,e_y - b\,e_z, \end{aligned}\right\} \tag{1.2}$$

[1] Pecora, L. M., and Carroll, T. L., *Synchronization in chaotic systems.* Phys. Rev. Lett. **64:**821, (1990).

[2] This equation replaces the first equation $\dot{x}_r = \sigma\,(y_r - x_r)$ in a Lorenz system for $(x_r,\, y_r,\, z_r)$. Then x is used to replace x_r in the other two equations. The Lorenz system constants are σ, r, b.

[3] He, R., and Vaidya, P. G., *Analysis and synthesis of synchronous periodic and chaotic systems.* Phys. Rev. A, **46:**7387 (1992).

where $e_x = x - x_r$, $e_y = y - y_r$, and $e_z = z - z_r$.

B. Show that $V = (e_y)^2 + (e_z)^2$ is a Liapunov function.

C. What do you conclude?

2 Hill equation problem #04 (with damping)

Statement: Hill equation problem #04 (with damping)

Let $\mathcal{S} = \mathcal{S}(\xi)$ be a periodic (of period 2π) function — i.e.: $\mathcal{S}(\xi + 2\pi) = \mathcal{S}(\xi)$. Consider now the damped Hill equation problem

$$\ddot{x} + 2\nu\dot{x} + \left(k^2 + a^2\,\mathcal{S}(\omega t)\right)\,x = 0, \tag{2.1}$$

where ν, k, a, $\omega > 0$ are constants — note that the **coefficients period is** $T = \dfrac{2\pi}{\omega}$.

Problem **tasks:**

1. *Write the equations in the standard form* $\dot{X} = \mathcal{A}(\omega t)\,X$, where $\mathcal{A}$ is a 2×2 matrix with period 2π and X is a two-vector.
2. *Write the Floquet multipliers* λ_j *in terms of* $\alpha = \frac{1}{2}\mathrm{Tr}(R)$, *where* R *is the Floquet matrix.*
   Hint. $\Delta = \det(R)$ can be computed explicitly.
3. *Write the stability/instability condition in terms of* α.
4. *Find the function* $\alpha_0 = \lim_{a\to 0}\alpha$. Then use it to *identify the places, if any, where an instability may occur for* $0 < a \ll 1$. That is, the values $k = k_*$ such that, for $0 < a \ll 1$, instabilities can arise for k near k_* only.
   Hint. For a small instabilities only arise near k's where a Floquet multiplier satisfies $|\lambda_j| = 1$ for $a = 0$.
5. *Plot* α_0 *versus* k/ω, *with* ν/ω *fixed, in a graph that includes the neutral stability curves. Use the range* $0 \le k/\omega \le 5.1$ *and take* $\nu/\omega = 0.06, 0.20,\ 0.50$.
   The neutral stability curves are lines in the α-k plane such that: a Floquet multiplier satisfies $|\lambda_j| = 1$ when/where the graph of α intersects the curve. You should know these curves from item **3**.
   Hint: when solving item **4** you should find that α_0 is a function of k/ω and ν/ω only, while the neutral stability boundary depends on ν/ω only.
6. *Plot* α_0 *versus* k/ω, *with* ν *a function of* k, *in a graph that includes the neutral stability curves. Use the range* $0 \le k/\omega \le 5.1$ *and take* $\nu = 0.06\,k, 0.12\,k,\ 0.10\,\frac{k^2}{\omega}$.

THE END.