Chapter 7

Matrix Completion

Here we will give algorithms for the matrix completion problem, where we observe uniformly random entries of a low-rank, incoherent matrix M and we would like design efficient algorithms that exactly recover M.

7.1 Background

The usual motivation for studying the matrix completion problem comes from recommendation systems. To be concrete, consider the Netflix problem where we are given ratings $M_{i,j}$ that represent how user i rated movie j. We would like to use these ratings to make good recommendations to users, and a standard approach is to try to use our knowledge of some of the entries of M to fill in the rest of M.

Let us be more precise: There is an unknown matrix $M \in \mathbb{R}^{n \times m}$ whose rows represent users and whose columns represent movies in the example above. For each $(i, j) \in \Omega \subseteq [n] \times [m]$ we are given the value $M_{i,j}$. Our goal is to recover M exactly. Ideally, we would like to find the minimum rank matrix X that agrees with M on the observed entries $\{M_{i,j}\}_{(i,j)\in\Omega}$ however this problem is NP-hard. There are some now standard assumptions under which we will be able to give efficient algorithms for recovering M exactly:

(a) Ω is uniformly random

(b) The singular vectors of M are uncorrelated with the standard basis (such a matrix is called *incoherent* and we define this later)

In fact, we will see that there are efficient algorithms for recovering M exactly if $m \approx mr \log m$ where $m \geq n$ and $\text{rank}(M) \leq r$. This is similar to compressed sensing, where we were able to recover a k-sparse signal x from $O(k \log n/k)$ linear measurements, which is much smaller than the dimension of x. Here too we can recover a low-rank matrix M from a number of observations that is much smaller than the dimension of M.

Let us examine the assumptions above. The assumption that should give us pause is that Ω is uniformly random. This is somewhat unnatural since it would be more believable if the probability we observe $M_{i,j}$ depends on the value itself. Alternatively, a user should be more likely to rate a movie *if he actually liked it.*

In order to understand the second assumption, suppose Ω is indeed uniformly random. Consider

$$M = \Pi \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right] \Pi^T$$

where Π is a uniformly random permutation matrix. M is low-rank, but unless we observe all of the ones along the diagonal, we will not be able to recover M uniquely. Indeed, the singular vectors of M contain some of the standard basis vectors; but if we were to assume that the singular vectors of M are incoherent with respect to the standard basis, we could avoid the above problem.

Definition 7.1.1 *The coherence μ of a subspace $U \subseteq \mathbb{R}^n$ of dimension $dim(u) = r$ is*

$$\frac{n}{r} \max_i \|P_U e_i\|^2,$$

where P_U denotes the orthogonal projection onto U, and e_i is the standard basis element.

It is easy to see that if we choose U uniformly at random, then $\mu(U) = \widetilde{O}(1)$. Also we have that $1 \leq \mu(U) \leq n/r$ and the upper bound is attained if U contains any e_i. We can now see that if we set U to be the top singular vectors of the above example, then U has high coherence. We will need the following conditions on M:

(a) Let $M = U\Sigma V^T$, then $\mu(U), \mu(V) \leq \mu_0$.

(b) $\|UV^T\|_\infty \leq \frac{\mu_1\sqrt{r}}{n}$, where $\|\cdot\|_\infty$ denotes the maximum absolute value of any entry.

The main result of this chapter is:

Theorem 7.1.2 *Suppose Ω is chosen uniformly at random. Then there is a polynomial time algorithm to recover M exactly that succeeds with high probability if*

$$m \geq \max(\mu_1^2, \mu_0) r(n+m) \log^2(n+m)$$

The algorithm in the theorem above is based on a convex relaxation for the rank of a matrix called the *nuclear norm*. We will introduce this in the next section, and establish some of its properties but one can think of it as an analogue to the ℓ_1 minimization approach that we used in compressed sensing. This approach was first introduced in Fazel's thesis [58], and Recht, Fazel and Parrilo [104] proved that this approach exactly recovers M in the setting of *matrix sensing*, which is related to the problem we consider here.

In a landmark paper, Candes and Recht [33] proved that the relaxation based on nuclear norm also succeeds for matrix completion and introduced the assumptions above in order to prove that their algorithm works. There has since been a long line of work improving the requirements on m, and the theorem above and our exposition will follow a recent paper of Recht [103] that greatly simplifies the analysis by making use of matrix analogues of the Bernstein bound and using these in a procedure now called *quantum golfing* that was first introduced by Gross [67].

Remark 7.1.3 *We will restrict to $M \in \mathbb{R}^{n\times n}$ and assume $\mu_0, \mu_1 = \widetilde{O}(1)$ in our analysis, which will reduce the number of parameters we need to keep track of. Also let $m = n$.*

7.2 Nuclear Norm

Here we introduce the nuclear norm, which will be the basis for our algorithms for matrix completion. We will follow a parallel outline to that of compressed sensing. In particular, a natural starting point is the optimization problem:

$$(P0) \qquad \min \operatorname{rank}(X) \text{ s.t. } X_{i,j} = M_{i,j} \text{ for all } (i,j) \in \Omega$$

This optimization problem is NP-hard. If $\sigma(X)$ is the vector of singular values of X then we can think of the rank of X equivalently as the sparsity of $\sigma(X)$. Recall, in compressed sensing we faced a similar obstacle: finding the sparsest solution to a system of linear equations is also NP-hard, but we instead considered the ℓ_1 relaxation and proved that under various conditions this optimization problem recovers the sparsest solution. Similarly it is natural to consider the ℓ_1-norm of $\sigma(X)$ which is called the nuclear norm:

Definition 7.2.1 *The nuclear norm of X denoted by $\|X\|_*$ is $\|\sigma(X)\|_1$.*

We will instead solve the convex program:

$$(P1) \qquad \min \|X\|_* \text{ s.t. } X_{i,j} = M_{i,j} \text{ for all } (i,j) \in \Omega$$

and our goal is to prove conditions under which the solution to $(P1)$ is exactly M. Note that this is a convex program because $\|X\|_*$ is a norm, and there are a variety of efficient algorithms to solve the above program.

In fact, for our purposes a crucial notion is that of a *dual norm*. We will not need this concept in full-generality, so we state it for the specific case of the nuclear norm. This concept gives us a method to lower bound the nuclear norm of a matrix:

Definition 7.2.2 *Let* $\langle X, B\rangle = \sum_{i,j} X_{i,j}B_{i,j} = trace(X^T B)$ *denote the matrix inner-product.*

Lemma 7.2.3 $\|X\|_* = \max_{\|B\|\leq 1}\langle X, B\rangle$.

To get a feel for this, consider the special case where we restrict X and B to be diagonal. Moreover let $X = \text{diag}(x)$ and $B = \text{diag}(b)$. Then $\|X\|_* = \|x\|_1$ and the constraint $\|B\| \leq 1$ (the spectral norm of B is at most one) is equivalent to $\|b\|_\infty \leq 1$. So we can recover a more familiar characterization of vector norms in the special case of diagonal matrices:

$$\|x\|_1 = \max_{\|b\|_\infty\leq 1} b^T x$$

Proof: We will only prove one direction of the above lemma. What B should we use to certify the nuclear norm of X. Let $X = U_X\Sigma_X V_X^T$, then we will choose $B = U_X V_X^T$. Then

$$\langle X, B\rangle = \text{trace}(B^T X) = \text{trace}(V_X U_X^T U_X \Sigma_X V_X^T) = \text{trace}(V_X\Sigma_X V_X^T) = \text{trace}(\Sigma_X) = \|X\|_*$$

where we have used the basic fact that $\text{trace}(ABC) = \text{trace}(BCA)$. Hence this proves $\|X\|_* \leq \max_{\|B\|\leq 1}\langle X, B\rangle$, and the other direction is not much more difficult (see e.g. [74]). ■

How can we show that the solution to $(P1)$ is M? Our basic approach will be a proof by contradiction. Suppose not, then the solution is $M + Z$ for some Z that is supported in $\overline{\Omega}$. Our goal will be to construct a matrix B of spectral norm at most one for which

$$\|M + Z\|_* \geq \langle M + Z, B\rangle > \|M\|_*$$

Hence $M + Z$ would not be the optimal solution to $(P1)$. This strategy is similar to the one in compressed sensing, where we hypothesized some other solution w that differs from x by a vector y in the kernel of the sensing matrix A. We used geometric properties of $\ker(A)$ to prove that w has strictly larger ℓ_1 norm than x. However the proof here will be more involved since our strategy is to construct B above based on Z (rather than relying on some geometry property of A that holds regardless of what y is).

Let us introduce some basic projection operators that will be crucial in our proof. Recall, $M = U\Sigma V^T$, let $u_1, \ldots, u_r$ be columns of U and let $v_1, \ldots, v_r$ be columns of V. Choose $u_{r+1}, \ldots, u_n$ so that $u_1, \ldots, u_n$ form an orthonormal basis for all of $\mathbb{R}^n$ – i.e. $u_{r+1}, \ldots, u_n$ is an arbitrary orthonormal basis of $U^\perp$. Similarly choose $v_{r+1}, \ldots, v_n$ so that $v_1, \ldots, v_n$ form an orthonormal basis for all of $\mathbb{R}^n$. We will be interested in the following linear spaces over matrices:

Definition 7.2.4 $T = span\{u_i v_j^T \mid 1 \leq i \leq r \ or \ 1 \leq j \leq r \ or \ both\}$.

Then $T^\perp = \text{span}\{u_i v_j^T \text{ s.t. } r+1 \leq i, j \leq n\}$.. We have $\dim(T) = r^2 + 2(n-r)r$ and $\dim(T^\perp) = (n-r)^2$. Moreover we can define the linear operators that project into T and $T^\perp$ respectively:

$$P_{T^\perp}[Z] = \sum_{i,j=r+1}^{n} \langle Z, u_i v_j^T \rangle \cdot U_i v_j^T = P_{U^\perp} Z P_{V^\perp}.$$

And similarly

$$P_T[Z] = \sum_{(i,j)\in[n]\times[n]-[r+1,n]\times[r+1,n]} \langle Z, u_i v_j^T \rangle \cdot u_i v_j^T = P_U Z + Z P_V - P_U Z P_V.$$

We are now ready to describe the outline of the proof of Theorem 7.1.2. The proof will be based on:

(a) We will assume that a certain helper matrix Y exists, and show that this is enough to imply $\|M + Z\|_* > \|M\|_*$ for any Z supported in Ω

(b) We will construct such a Y using quantum golfing [67].

Part *(a)*

Here we will state the conditions we need on the helper matrix Y and prove that if such a Y exists, then M is the solution to $(P1)$. We require that Y is supported in Ω and

(a) $\|P_T(Y) - UV^T\|_F \leq \sqrt{r/8n}$

(b) $\|P_{T^\perp}(Y)\| \leq 1/2$.

We want to prove that for any Z supported in $\overline{\Omega}$, $\|M+Z\|_* > \|M\|_*$. Recall, we want to find a matrix B of spectral norm at most one so that $\langle M+Z, B\rangle > \|M\|_*$. Let $U_\perp$ and $V_\perp$ be singular vectors of $P_{T^\perp}[Z]$. Then consider

$$B = \begin{bmatrix} U & U_\perp \end{bmatrix} \cdot \begin{bmatrix} V^T \\ V_\perp^T \end{bmatrix} = UV^T + U_\perp V_\perp^T.$$

Claim 7.2.5 $\|B\| \leq 1$

Proof: By construction $U^T U_\perp = 0$ and $V^T V_\perp = 0$ and hence the above expression for B is its singular value decomposition, and the claim now follows. ■

Hence we can plug in our choice for B and simplify:

$$\begin{aligned}\|M+Z\|_* &\geq \langle M+Z, B\rangle \\ &= \langle M+Z, UV^T + U_\perp V_\perp^T\rangle \\ &= \underbrace{\langle M, UV^T\rangle}_{\|M\|_*} + \langle Z, UV^T + U_\perp V_\perp^T\rangle\end{aligned}$$

where in the last line we used the fact that M is orthogonal to $U_\perp V_\perp^T$. Now using the fact that Y and Z have disjoint supports we can conclude:

$$\|M+Z\|_* \geq \|M\|_* + \langle Z, UV^T + U_\perp V_\perp^T - Y\rangle$$

Therefore in order to prove the main result in this section it suffices to prove that $\langle Z, UV^T + U_\perp V_\perp^T - Y\rangle > 0$. We can expand this quantity in terms of its projection onto T and $T^\perp$ and simplify as follows:

$$\begin{aligned}\|M+Z\|_* - \|M\|_* &\geq \langle P_T(Z), P_T(UV^T + U_\perp V_\perp^T - Y)\rangle + \langle P_{T^\perp}(Z), P_{T^\perp}(UV^T + U_\perp V_\perp^T - Y)\rangle \\ &\geq \langle P_T(Z), UV^T - P_T(Y)\rangle + \langle P_{T^\perp}(Z), U_\perp V_\perp^T - P_{T^\perp}(Y)\rangle \\ &\geq \langle P_T(Z), UV^T - P_T(Y)\rangle + \|P_{T^\perp}(Z)\|_* - \langle P_{T^\perp}(Z), P_{T^\perp}(Y)\rangle\end{aligned}$$

where in the last line we used the fact that $U_\perp$ and $V_\perp$ are the singular vectors of $P_{T^\perp}[Z]$ and hence $\langle U_\perp V_\perp^T, P_{T^\perp}[Z]\rangle = \|P_{T^\perp}[Z]\|_*$.

Now we can invoke the properties of Y that we have assumed in this section, to prove a lower bound on the right hand side. By property (a) of Y, we have that $\|P_T(Y) - UV^T\|_F \leq \sqrt{\frac{r}{2n}}$. Therefore, we know that the first term $\langle P_T(Z), UV^T - P_T(Y)\rangle \geq -\sqrt{\frac{r}{8n}}\|P_T(Z)\|_F$. By property (b) of Y, we know the operator norm of $P_T^{\perp}(Y)$ is at most $1/2$. Therefore the third term $\langle P_{T^\perp}(Z), P_{T^\perp}(Y)\rangle$ is at most $\frac{1}{2}\|P_{T^\perp}(Z)\|_*$. Hence

$$\|M+Z\|_* - \|M\|_* \geq -\sqrt{\frac{r}{8n}}\|P_T(Z)\|_F + \frac{1}{2}\|P_{T^\perp}(Z)\|_* \overset{?}{>} 0$$

We will show that with high probability over the choice of Ω that the inequality does indeed hold. We defer the proof of this last fact, since it and the construction of the helper matrix Y will both make use of the matrix Bernstein inequality which we present in the next section.

7.3 Quantum Golfing

What remains is to construct a helper matrix Y and prove that with high probability over Ω, for any matrix Z supported in $\overline{\Omega}$ that $\|P_{T^\perp}(Z)\|_* > \sqrt{\frac{r}{2n}}\|P_T(Z)\|_F$ to complete the proof we started in the previous section. We will make use of an approach introduced by Gross [67] and we will follow the proof of Recht in [103] where the strategy is to construct Y iteratively. In each phase, we will invoke concentration results for matrix valued random variables to prove that the error part of Y decreases geometrically and we make rapid progress in constructing a good helper matrix.

First we will introduce the key concentration result that we will apply in several settings. The following matrix valued Bernstein inequality first appeared in the work of Ahlswede and Winter related to quantum information theory [6].

Theorem 7.3.1 (Non-commutative Bernstein Inequality) *Let $X_1 \dots X_l$ be independent mean 0 matrices of size $d \times d$. Let $\rho_k^2 = \max\{\|\mathbb{E}[X_k X_k^T]\|, \|\mathbb{E}[X_k^T X_k]\|\}$ and suppose $\|X_k\| \leq M$ almost surely. Then for $\tau > 0$,*

$$\Pr\left[\Big\|\sum_{k=1}^{l} X_k\Big\| > \tau\right] \leq 2d\exp\left\{\frac{-\tau^2/2}{\sum_k \rho_k^2 + M\tau/3}\right\}$$

If $d = 1$ this is the standard Bernstein inequality. If $d > 1$ and the matrices X_k are diagonal then this inequality can be obtained from the union bound and the standard Bernstein inequality again. However to build intuition, consider the following toy problem. Let u_k be a random unit vector in $\mathbb{R}^d$ and let $X_k = u_k u_k^T$. Then it is easy to see that $\rho_k^2 = 1/d$. How many trials do we need so that $\sum_k X_k$ is close to the identity (after scaling)? We should expect to need $\Theta(d \log d)$ trials; this is even true if u_k is drawn uniformly at random from the standard basis vectors $\{e_1 \dots e_d\}$ due to the coupon collector problem. Indeed, the above bound corroborates our intuition that $\Theta(d \log d)$ is necessary and sufficient.

Now we will apply the above inequality to build up the tools we will need to finish the proof.

Definition 7.3.2 *Let R_Ω be the operator that zeros out all the entries of a matrix except those in Ω.*

Lemma 7.3.3 *If Ω is chosen uniformly at random and $m \geq nr \log n$ then with high probability*

$$\frac{n^2}{m}\left\|P_T R_\Omega P_T - \frac{m}{n^2} P_T\right\| < \frac{1}{2}$$

Remark 7.3.4 *Here we are interested in bounding the operator norm of a linear operator on matrices. Let T be such an operator, then $\|T\|$ is defined as*

$$\max_{\|Z\|_F \leq 1} \|T(Z)\|_F$$

We will explain how this bound fits into the framework of the matrix Bernstein inequality, but for a full proof see [103]. Note that $\mathbb{E}[P_T R_\Omega P_T] = P_T \, \mathbb{E}[R_\Omega] P_T = \frac{m}{n^2} P_T$ and so we just need to show that $P_T R_\Omega P_T$ does not deviate too far from its expectation. Let $e_1, e_2, \ldots, e_d$ be the standard basis vectors. Then we can expand:

$$\begin{aligned}
P_T(Z) &= \sum_{a,b} \langle P_T(Z), e_a e_b^T \rangle e_a e_b^T \\
&= \sum_{a,b} \langle P_T(Z), e_a e_b^T \rangle e_a e_b^T \\
&= \sum_{a,b} \langle Z, P_T(e_a e_b^T) \rangle e_a e_b^T
\end{aligned}$$

Hence $R_\Omega P_T(Z) = \sum_{(a,b)\in\Omega} \left\langle Z, P_T(e_a e_b^T) \right\rangle e_a e_b^T$ and finally we conclude that

$$P_T R_\Omega P_T(Z) = \sum_{(a,b)\in\Omega} \left\langle Z, P_T(e_a e_b^T) \right\rangle P_T(e_a e_b^T)$$

We can think of $P_T R_\Omega P_T$ as the sum of random operators of the form $\tau_{a,b} : Z \to \left\langle Z, P_T(e_a e_b^T) \right\rangle P_T(e_a e_b^T)$, and the lemma follows by applying the matrix Bernstein inequality to the random operator $\sum_{(a,b)\in\Omega} \tau_{a,b}$.

We can now complete the deferred proof of part (a):

Lemma 7.3.5 *If Ω is chosen uniformly at random and $m \geq nr \log n$ then with high probability for any Z supported in $\overline{\Omega}$ we have*

$$\|P_{T^\perp}(Z)\|_* > \sqrt{\frac{r}{2n}}\|P_T(Z)\|_F$$

Proof: Using Lemma 7.3.3 and the definition of the operator norm (see the remark) we have

$$\left\langle Z, P_T R_\Omega P_T Z - \frac{m}{n^2} P_T Z \right\rangle \geq -\frac{m}{2n^2}\|Z\|_F^2$$

Furthermore we can upper bound the left hand side as:

$$\begin{aligned}\langle Z, P_T R_\Omega P_T Z\rangle = \langle Z, P_T R_\Omega^2 P_T Z\rangle &= \|R_\Omega(Z - P_{T^\perp}(Z))\|_F^2 \\ &= \|R_\Omega(P_{T^\perp}(Z))\|_F^2 \leq \|P_{T^\perp}(Z)\|_F^2\end{aligned}$$

where in the last line we used that Z is supported in $\overline{\Omega}$ and so $R_\Omega(Z) = 0$. Hence we have that

$$\|P_{T^\perp}(Z)\|_F^2 \geq \frac{m}{n^2}\|P_T(Z)\|_F^2 - \frac{m}{2n^2}\|Z\|_F^2$$

We can use the fact that $\|Z\|_F^2 = \|P_{T^\perp}(Z)\|_F^2 + \|P_T(Z)\|_F^2$ and conclude $\|P_{T^\perp}(Z)\|_F^2 \geq \frac{m}{4n^2}\|P_T(Z)\|_F^2$. We can now complete the proof of the lemma

$$\begin{aligned}\|P_{T^\perp}(Z)\|_*^2 \geq \|P_{T^\perp}(Z)\|_F^2 &\geq \frac{m}{4n^2}\|P_T(Z)\|_F^2 \\ &> \frac{r}{2n}\|P_T(Z)\|_F^2\end{aligned}$$

■

All that remains is to prove that the helper matrix Y that we made use of actually does exists (with high probability). Recall that we require that Y is supported in Ω and $\|P_T(Y) - UV^T\|_F \leq \sqrt{r/8n}$ and $\|P_{T^\perp}(Y)\| \leq 1/2$. The basic idea is to break up Ω into disjoint sets $\Omega_1, \Omega_2, \ldots \Omega_p$, where $p = \log n$ and use each set of observations to make progress on the remained $P_T(Y) - UV^T$. More precisely, initialize $Y_0 = 0$ in which case the remainder is $W_0 = UV^T$. Then set

$$Y_{i+1} = Y_i + \frac{n^2}{m} R_{\Omega_{i+1}}(W_i)$$

and update $W_{i+1} = UV^T - P_T(Y_{i+1})$. It is easy to see that $\mathbb{E}[\frac{n^2}{m} R_{\Omega_{i+1}}] = I$. Intuitively this means that at each step $Y_{i+1} - Y_i$ is an unbiased estimator for W_i and so we should expect the remainder to decrease quickly (here we will rely on the concentration bounds we derived from the non-commutative Bernstein inequality). Now we can explain the nomenclature *quantum golfing*; at each step, we hit our golf ball in the direction of the hole but here our target is to approximate the matrix UV^T which for various reasons is the type of question that arises in quantum mechanics.

It is easy to see that $Y = \sum_i Y_i$ is supported in Ω and that $P_T(W_i) = W_i$ for all i. Hence we can compute

$$\begin{aligned}\|P_T(Y_i) - UV^T\|_F &= \left\|W_{i-1} - P_T \frac{n^2}{m} R_{\Omega_i} W_{i-1}\right\|_F = \left\|P_T W_{i-1} - P_T \frac{n^2}{m} R_{\Omega_i} P_T W_{i-1}\right\|_F \\ &\leq \frac{n^2}{m} \left\|P_T R_\Omega P_T - \frac{m}{n^2} P_T\right\| \leq \frac{1}{2} \|W_{i-1}\|_F\end{aligned}$$

where the last inequality follows from Lemma 7.3.3. Therefore the Frobenius norm of the remainder decreases geometrically and it is easy to guarantee that Y satisfies condition (a).

The more technically involved part is showing that Y also satisfies condition (b). However the intuition is that $\|P_{T^\perp}(Y_1)\|$ is itself not too large, and since the norm of the remainder W_i decreases geometrically we should expect that $\|P_{T^\perp}(Y_i)\|$ does too and so most of the contribution to

$$\|P_{T^\perp}(Y)\| \leq \sum_i \|P_{T^\perp}(Y_i)\|$$

comes from the first term. For the full details see [103]. This completes the proof that computing the solution to the convex program indeed finds M *exactly*, provided that M is incoherent and $|\Omega| \geq \max(\mu_1^2, \mu_0) r(n+m) \log^2(n+m)$.