18.433 Combinatorial Optimization	Oct 30, Nov 4

Lecture The Ellipsoid Algorithm

Oct 30, Nov 4 — *Lecturer: Santosh Vempala*

1 The Algorithm for Linear Programs

Problem 1. Given a polyhedron P, written as $Ax \le b$, find a point in P.

Before tackling this problem, we begin with some definitions. A real symmetric matrix A with the property that $x^T Ax > 0$ for all $x \neq 0$ is called *positive definite*. If A is positive definite, then there exists an invertible matrix P, such that $A = P^T P$. Let D be a positive definite matrix and consider the ellipsoid $\text{Ell}(D, z) = \{x : (x - z)^T D^{-1}(x - z) \le 1\}$. Let ν be the maximum number of bits required to describe a vertex of P and set $R = 2^\nu$. To solve Problem 1 we apply the following algorithm:

The Ellipsoid Algorithm
Start with the ellipsoid $E_0 = \text{Ell}(R^2 I, 0)$.

At the i^{th} iteration, check whether z_i is in P.

- YES. Output z_i as the feasible point.

- NO. Find a constraint for P, $a_k \cdot x \le b_k$, violated by z_i. Recurse on E_{i+1}, the minimum volume ellipsoid containing $E_i \cap \{x \,|\, a_k \cdot x \le a_k \cdot z_i\}$.

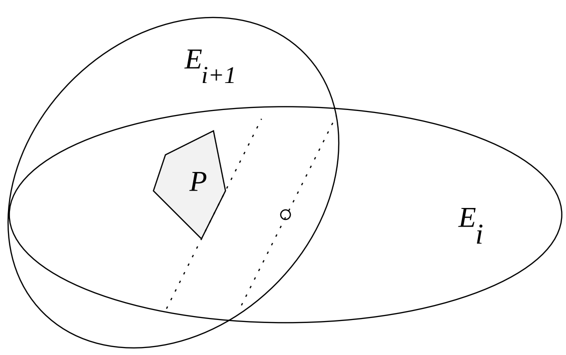

Figure 1: One cycle of the algorithm

Observe that the algorithm halts when a point z_i is found to be within P. It must halt, since at any step i, P is a subset of E_i, and we will see that after each step, the volume of E_{i+1} has decreased by an appreciable amount. For some value of i, the volume of E_i will be smaller than the volume of P, so the algorithm must halt before reaching this point. In the next section we show that the algorithm can actually be implemented in polynomial time.

2 The Time Bound

Lemma 1. *The minimum volume ellipsoid containing* $\mathrm{Ell}(D, z) \cap \{x \mid a \cdot x \leq a \cdot z\}$ *is exactly* $E' = \mathrm{Ell}(D', z')$, *where*

$$z' = z - \frac{1}{n+1} \frac{Da}{\sqrt{a^T Da}} \tag{1}$$

and

$$D' = \frac{n^2}{n^2 - 1} \left(D - \frac{2}{n+1} \frac{Daa^T D}{a^T Da} \right) \tag{2}$$

and

$$\frac{\mathrm{vol}(E')}{\mathrm{vol}(E)} \leq e^{\frac{-1}{2n+2}} \tag{3}$$

Sketch of proof : First, note that $\mathrm{Ell}(A, 0)$ can be obtained from $\mathrm{Ell}(I, 0)$ (the unit ball) using the transformation $y = Bx$, where $A = B^T B$. To see this, consider the following:

$$\begin{aligned} x^T x &\leq 1 \\ y^T (B^{-1})^T (B^{-1}) y &= x^T x \\ y^T (B^{-1})^T (B^{-1}) y &\leq 1 \\ y^T A^{-1} y &\leq 1 \end{aligned}$$

where the first and last equations define the unit ball and $\mathrm{Ell}(A, 0)$, respectively.

Now, first we will prove the results (1) and (2) for the special case of the unit ball, $E = \mathrm{Ell}(I, 0)$. In this case, (1) reduces to

$$z' = z - \frac{1}{n+1} \frac{a}{\sqrt{a^T a}}$$

and (2) reduces to

$$D' = \frac{n^2}{n^2 - 1} (I - \frac{2}{n+1} \frac{aa^T}{a^T a})$$

Since E is a ball, we can rotate a without affecting anything. So, assume $a = [1, 0, \cdots, 0]^T$. Then, $z' = [-1/(n+1), 0, \cdots, 0]^T$.

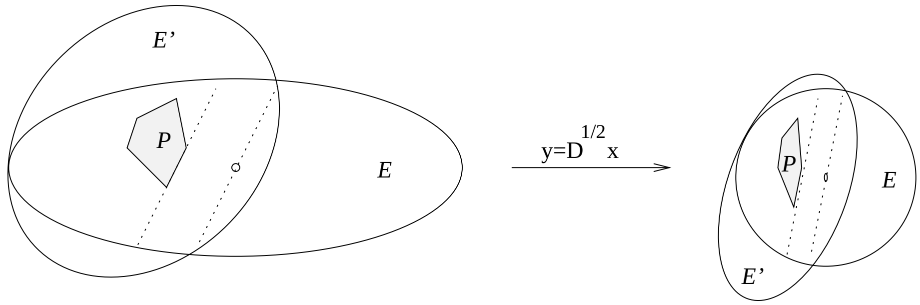

Figure 2: Transform space to take E to a ball

$$D' = \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1}\begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}\right)$$

$$= \frac{n^2}{n^2-1}\begin{bmatrix} 1-\frac{2}{n+1} & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \end{bmatrix}$$

The simplified statements for z' and D' can be proved by calculus. The general case is then proved by applying a transformation of A to the unit ball (B^{-1} above; the transformation scales the volume of a convex set by the factor $\det(B)$).

Assuming (1) and (2), we can now prove (3). observe that:

$$\frac{\mathrm{vol}(\mathrm{Ell}(D', z'))}{\mathrm{vol}(\mathrm{Ell}(D, z))} = \frac{\mathrm{vol}(\mathrm{Ell}(I, 0))}{\mathrm{vol}(\mathrm{Ell}(I, 0))}\frac{\sqrt{\det(D')}}{\sqrt{\det(D)}}$$

We transform space to take E to the unit ball. The transformed E' is still the minimum ellipsoid containing half of E.

Then, assuming $D = I$, we have $\sqrt{\det(D')} = \text{vol}(E')/\text{vol}(E)$. Now we can use (2).

$$D' = \frac{n^2}{n^2-1}\left(I - \frac{2}{n+1}\begin{bmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}\right)$$

The determinant of this matrix is

$$\det(D') = \left(\frac{n^2}{n^2-1}\right)^n \left(1 - \frac{2}{n+1}\right)$$

Hence,

$$\begin{aligned}
\frac{\text{vol}(E')}{\text{vol}(E)} &= \left(\frac{n^2}{n^2-1}\right)^{n/2}\left(\frac{n-1}{n+1}\right)^{1/2} \\
&= \left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}} \frac{n}{(n-1)^{1/2}(n+1)^{1/2}} \frac{(n-1)^{1/2}}{(n+1)^{1/2}} \\
&= \left(1 + \frac{1}{n^2-1}\right)^{\frac{n-1}{2}} \left(1 - \frac{1}{n+1}\right) \\
&\leq e^{\frac{1}{(n-1)(n+1)}\frac{(n-1)}{2}} e^{\frac{-1}{n+1}} = e^{\frac{-1}{2(n+1)}}
\end{aligned}$$

(using $e^x \geq 1 + x$). $\square$

We need to calculate how small P can be in order to obtain a bound on the number of times we shrink E'. We will see that $\text{vol}(P) \geq 2^{-2n\nu}$ by finding a simplex inside P. Clearly the volume of the simplex will be less than or equal to the volume of P.

Now there exist $n+1$ affinely independent verticies of P, say $x_0, x_1, \ldots, x_n$.

$$\text{vol}(\text{conv}(x_0, \ldots, x_n)) = \frac{1}{n!}\left|\det\begin{bmatrix} 1 & 1 & & 1 \\ & & \cdots & \\ x_0 & x_1 & & x_n \end{bmatrix}\right|$$

A vertex x_i is a solution to a subset C_i of rows of $Ax \leq b$. We can solve for it using Cramer's Rule, $x_{ij} = \frac{\det(C_{ij})}{\det(C_j)}$, where C_{ij} is the matrix C_i with the i^{th} column replaced by b restricted to the relevant rows for C_i. So,

$$\text{vol}(\text{conv}(x_0, \ldots, x_n)) = \frac{1}{n!}\left|\det\begin{bmatrix} 1 & 1 & \\ \frac{\det C_{11}}{\det C_1} & \frac{\det C_{12}}{\det C_2} & \cdots \\ \frac{\det C_{21}}{\det C_1} & \frac{\det C_{22}}{\det C_2} & \\ \vdots & & \ddots \end{bmatrix}\right|$$

Pulling out the denominators, we see that

$$\frac{1}{n!}\left|\det\left(\begin{bmatrix} \det C_1 & \det C_2 & & \\ \det C_{11} & \det C_{12} & & \\ & & \ddots & \\ & & & \det C_{nn} \end{bmatrix}\begin{bmatrix} \frac{1}{\det C_1} & & & \\ & \frac{1}{\det C_2} & & \\ & & \ddots & \\ & & & \frac{1}{\det C_n} \end{bmatrix}\right)\right|$$

$$\geq \frac{1}{n!}\frac{1}{\det(C_1)\det(C_2)\cdots\det(C_n)}$$

As $\det C_i \leq 2^{\nu}$, we have $\mathrm{vol}(\mathrm{conv}(x_0,\ldots,x_n)) \geq n^{-n}(2^{-\nu})^n \geq 2^{-2n\nu}$. After i steps, $\mathrm{vol}(E_i) \leq (2R)^n e^{\frac{-i}{2n+2}}$. We stop before $\mathrm{vol}(E_i) < \mathrm{vol}(P)$. Thus,

$$2^{(\nu+1)n}e^{\frac{-i}{2n+2}} < 2^{-2n\nu}$$

which means we stop when $i = \mathrm{O}(n^2\nu)$. Recall that ν was less than the number of bits required to write down any $n \times n$ subset of $\{A, b\}$, plus $\log n$ bits. So, the the number of iterations is $\mathrm{O}(n^2\langle C, d\rangle)$. If we use L-bit numbers, then $\langle C, d\rangle = \mathrm{O}(n^2 L)$. To check the validity of a point, we must check each constraint of P, taking $\mathrm{O}(mn)$ time. This dominates the time required to calculate the minimum ellipsoid. So the total time required to complete the algorithm is at most $\mathrm{O}(mn^5 L)$.