18.433 Combinatorial Optimization

The Matching Polytope: Bipartite Graphs

September 18 *Lecturer: Santosh Vempala*

A matching M corresponds to a vector $\chi^M = (0,0,1,1,0...0)$ of size $|E|$ where χ_e^M is 1 if $e \in M$ and 0 if $e \notin M$. Let $\mathcal{M}$ be the convex hull of all vectors corresponding to matchings.

$$\mathcal{M} = conv\{x = \chi^M \mid M \text{ is a matching}\}$$

and the resulting relaxation of the integral constraints:

$$P = \{x \mid \quad x_e \geq 0 \quad \forall e \in E, \sum_{e \in \delta(v)} x_e \leq 1 \; \forall v \in V\}$$

We now claim that $\mathcal{M} \subseteq P$. Note that $\mathcal{M}$ is convex. Clearly P is also convex (since the constraints are linear). Now $\mathcal{M} = conv(x_1 \ldots x_N)$ where $x_1 \ldots x_N \in P$. Since P is convex, $conv(x_1 \ldots x_N) \subseteq P$.

When is $\mathcal{M} = P$? It's not true in general.

All vertices of $\mathcal{M}$ are $0-1$ points. If the vertices of P were integral, then they must be either 0 or 1. But then they must be a matching. So when are the vertices of P integral? They're the solutions to n independent equations and also the points that can't be expressed as a convex combination of two other points.

Theorem 1. *If G is bipartite, then $P = \mathcal{M}$.*

Proof I. Suppose this isn't the case. Then P has a vertex that is not integral (implied by $P \neq \mathcal{M}$). So take a non-integral vertex x with the fewest non-integral elements. Let $G_x = (V, E_x)$, the graph with only fractional valued edges. Suppose there is a cycle. It must be an even length cycle because the graph is bipartite. Let ϵ to be $min(a, 1-b)$ where a (b) is the minimum (maximum) value of edges in the cycle. Add ϵ to alternating edges in the cycle and subtract ϵ from the other (also alternating) edges to get $x' = x + \epsilon z$ where $z = (1, -1, 1, -1 \ldots 1, -1)$ along edges in the cycle (zero on other edges). So x' still satisfies the constraints in P and hence $x' \in P$. Now let $x'' = x - \epsilon z$. Note that $x'' \in P$ and $x = \frac{1}{2}(x' + x'')$. Why is x not a vertex? x is a convex combination of two other points in P, so it can't be a vertex. If there are no cycles, then we can apply the same idea along some path. Hence the vertices of P are integral and the theorem follows. □

We note that the fact that G is bipartite prevents odd cycles; the trick in Theorem 1 cannot be done on an odd cycle since the vertex constraint for one of the vertices on the cycle may become violated for x' or x''.

Next we give an alternative proof of Theorem 1 based on a different concept of what a vertex of a polytope is.

Proof II. We will prove that if G is bipartite, then the vertices of P are integral. For the purpose of this proof, we will consider a vertex of a polytope as a solution to n (where n is the dimension of the space) linearly independent hyperplanes (i.e. facets). This implies that it must satisfy n linearly independent inequalities (constraints) as equalities.

We can describe P by using a matrix inequality to represent the given constraint conditions.

$$P = \{x \mid Ax \leq b\}$$

If y is a vertex of the polytope defined by the constraint matrix A then there exists a subset of rows A_1 of A, and a corresponding subset $b_1 \subseteq b$ such that

$$A_1 y = b_1$$

and

$$det(A_1) \neq 0$$

Given such a system, we can easily solve for y using Cramer's rule.

$$y_i = \frac{det(A_1^{(i)})}{det(A_1)}$$

where $A_1^{(i)}$ is the matrix A_1 with the ith column replaced by b_1.

Our question is: when is the vertex integral? One sufficient condition is the following:

Observation 2. *The vertex solution y is integral if the numerator of the value above is integral and $det(A_i) = \pm 1$.*

Note that we need this for all $n \times n$ submatrices A_1 (with $det(A_1) \neq 0$). The first condition holds since A is an integral matrix and b is integral. To show the second condition holds we will make use of the following definition.

Definition 3. *A matrix is said to be totally unimodular if every square submatrix has a determinant of 0, 1 or -1.*

We will show that the constraint matrix (defined below) for the given polytope is indeed totally unimodular, which will give us our desired result.

The first part of the matrix comes from the second constraint set of P:

$$\sum_{e\in\delta(v)} x_e \leq 1$$

This simply gives us the vertex-edge adjacency matrix of the graph G, A_{adj}. A_{adj} has $|V|$ rows and $|E|$ columns, with each element being defined as follows:

$$a_{ve} = \begin{cases} 0 & \text{if } \; e \notin \delta(v) \\ 1 & \text{if } \; e \in \delta(v) \end{cases}$$

($\delta(v)$ is the set of neighbors of v)

We also wish to put the first set of constraints, $x_e \geq 0 \quad \forall e \in E$, into the form $A\mathbf{x} \leq b$. This we achieve by the use of the negative of the identity matrix.

Therefore, our constraint matrix, A, consists of two parts:

1. The $|V| \times |E|$ adjacency matrix on top.
2. The negative of the $|E| \times |E|$ identity matrix on the bottom.

Lemma 4. *If G is bipartite, then the constraint matrix is totally unimodular.*

Proof. Take any $k \times k$ submatrix Q of A. We will prove the result by induction on k.

For $k = 1$, this is true because each element of A is 0 or ± 1 by construction.

Now assume the lemma true for all $(k-1) \times (k-1)$ size submatrices.

Consider a matrix Q of size $k \times k$.

1. If any row in Q is all zero, then $|Q| = 0$ and we are done.
2. If any row in Q has only a single non-zero element that is ± 1, then we can expand around that 1 or -1 and use our induction hypothesis on the $(k-1) \times (k-1)$ size submatrix to deduce that $|Q| = 0$ or ± 1, and we are done.
3. If we then assume that every row in Q has more than one non-zero element, then Q must come entirely from the upper half of the constraint matrix, A_{adj}, since the lower half is the identity matrix. But if this is the case, then the bipartiteness of G

implies that we can partition the rows of Q into two sections corresponding to the partitioning of the vertices of G. However, if we sum the rows of each section, we will get the same vectors, because each edge from E touches exactly one vertex from each section and hence each column has a 1 in each section. This implies that Q is a dependent system and hence $|Q| = 0$.

We have shown that A is totally unimodular, which proves our lemma, and thus shows that the vertices of P are indeed integral, which proves that $P = \mathcal{M}$. □

A similar theorem holds for the perfect matching polytope $\mathcal{PM}(G)$, the convex hull of perfect matching of G. The only change we need to make to P is to replace the inequality at each vertex by an equality, i.e. the sum of the edges at each vertex is 1.

What about for non-bipartite graphs? Clearly the same constraints do not apply, because we can have a triangle with edge weights of $\frac{1}{2}$ on each side. In such a situation, $P(G) \neq \mathcal{M}(G)$ because there exist no perfect matchings of G, while the point $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in P(G)$.