Lecture 6

Gamma distribution, χ^2-distribution, Student t-distribution, Fisher F-distribution.

Gamma distribution. Let us take two parameters $\alpha > 0$ and $\beta > 0$. Gamma function $\Gamma(\alpha)$ is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx.$$

If we divide both sides by $\Gamma(\alpha)$ we get

$$1 = \int_0^\infty \frac{1}{\Gamma(\alpha)}x^{\alpha-1}e^{-x}dx = \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{-\beta y}dy$$

where we made a change of variables $x = \beta y$. Therefore, if we define

$$f(x|\alpha,\beta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

then $f(x|\alpha,\beta)$ will be a probability density function since it is nonnegative and it integrates to one.

Definition. The distribution with p.d.f. $f(x|\alpha,\beta)$ is called Gamma distribution with parameters α and β and it is denoted as $\Gamma(\alpha,\beta)$.

Next, let us recall some properties of gamma function $\Gamma(\alpha)$. If we take $\alpha > 1$ then using integration by parts we can write:

$$\begin{aligned}
\Gamma(\alpha) &= \int_0^\infty x^{\alpha-1}e^{-x}dx = \int_0^\infty x^{\alpha-1}d(-e^{-x}) \\
&= x^{\alpha-1}(-e^{-x})\Big|_0^\infty - \int_0^\infty (-e^{-x})(\alpha-1)x^{\alpha-2}dx \\
&= (\alpha-1)\int_0^\infty x^{(\alpha-1)-1}e^{-x}dx = (\alpha-1)\Gamma(\alpha-1).
\end{aligned}$$

Since for $\alpha = 1$ we have

$$\Gamma(1) = \int_0^\infty e^{-x} dx = 1$$

we can write

$$\Gamma(2) = 1 \cdot 1, \, \Gamma(3) = 2 \cdot 1, \, \Gamma(4) = 3 \cdot 2 \cdot 1, \, \Gamma(5) = 4 \cdot 3 \cdot 2 \cdot 1$$

and proceeding by induction we get that $\Gamma(n) = (n-1)!$

Let us compute the kth moment of gamma distribution. We have,

$$\begin{aligned}
\mathbb{E}X^k &= \int_0^\infty x^k \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{(\alpha+k)-1} e^{-\beta x} dx \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+k)}{\beta^{\alpha+k}} \underbrace{\int_0^\infty \frac{\beta^{\alpha+k}}{\Gamma(\alpha+k)} x^{\alpha+k-1} e^{-\beta x} dx}_{p.d.f. \text{ of } \Gamma(\alpha+k,\beta) \text{ integrates to } 1} \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+k)}{\beta^{\alpha+k}} = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\beta^k} = \frac{(\alpha+k-1)\Gamma(\alpha+k-1)}{\Gamma(\alpha)\beta^k} \\
&= \frac{(\alpha+k-1)(\alpha+k-2)\ldots\alpha\Gamma(\alpha)}{\Gamma(\alpha)\beta^k} = \frac{(\alpha+k-1)\cdots\alpha}{\beta^k}.
\end{aligned}$$

Therefore, the mean is

$$\mathbb{E}X = \frac{\alpha}{\beta}$$

the second moment is

$$\mathbb{E}X^2 = \frac{(\alpha+1)\alpha}{\beta^2}$$

and the variance

$$\mathrm{Var}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{(\alpha+1)\alpha}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2 = \frac{\alpha}{\beta^2}.$$

Below we will need the following property of Gamma distribution.

Lemma. *If we have a sequence of independent random variables*

$$X_1 \sim \Gamma(\alpha_1, \beta), \ldots, X_n \sim \Gamma(\alpha_n, \beta)$$

then $X_1 + \ldots + X_n$ *has distribution* $\Gamma(\alpha_1 + \ldots + \alpha_n, \beta)$

Proof. If $X \sim \Gamma(\alpha, \beta)$ then a moment generating function (m.g.f.) of X is

$$\begin{aligned}
\mathbb{E}e^{tX} &= \int_0^\infty e^{tx} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} dx = \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\beta-t)x} dx \\
&= \frac{\beta^\alpha}{(\beta-t)^\alpha} \underbrace{\int_0^\infty \frac{(\beta-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\beta-t)x} dx}.
\end{aligned}$$

The function in the last (underbraced) integral is a p.d.f. of gamma distribution $\Gamma(\alpha, \beta - t)$ and, therefore, it integrates to 1. We get,

$$\mathbb{E}e^{tX} = \Big(\frac{\beta}{\beta - t}\Big)^{\alpha}.$$

Moment generating function of the sum $\sum_{i=1}^n X_i$ is

$$\mathbb{E}e^{t\sum_{i=1}^n X_i} = \mathbb{E}\prod_{i=1}^n e^{tX_i} = \prod_{i=1}^n \mathbb{E}e^{tX_i} = \prod_{i=1}^n \Big(\frac{\beta}{\beta - t}\Big)^{\alpha_i} = \Big(\frac{\beta}{\beta - t}\Big)^{\sum \alpha_i}$$

and this is again a m.g.f. of Gamma distibution, which means that

$$\sum_{i=1}^n X_i \sim \Gamma\Big(\sum_{i=1}^n \alpha_i, \beta\Big).$$

□

χ^2_n-distribution. In the previous lecture we defined a χ^2_n-distribution with n degrees of freedom as a distribution of the sum $X_1^2 + \ldots + X_n^2$, where X_is are i.i.d. standard normal. We will now show that which χ^2_n-distribution coincides with a gamma distribution $\Gamma(\frac{n}{2}, \frac{1}{2})$, i.e.

$$\chi^2_n = \Gamma\Big(\frac{n}{2}, \frac{1}{2}\Big).$$

Consider a standard normal random variable $X \sim N(0,1)$. Let us compute the distribution of X^2. The c.d.f. of X^2 is given by

$$\mathbb{P}(X^2 \leq x) = \mathbb{P}(-\sqrt{x} \leq X \leq \sqrt{x}) = \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$$

The p.d.f. can be computed by taking a derivative $\frac{d}{dx}\mathbb{P}(X \leq x)$ and as a result the p.d.f. of X^2 is

$$\begin{aligned} f_{X^2}(x) &= \frac{d}{dx}\int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{x})^2}{2}} (\sqrt{x})' - \frac{1}{\sqrt{2\pi}} e^{-\frac{(-\sqrt{x})^2}{2}} (-\sqrt{x})' \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{x}} e^{-\frac{x}{2}} = \frac{1}{\sqrt{2\pi}} x^{\frac{1}{2}-1} e^{-\frac{x}{2}}. \end{aligned}$$

We see that this is p.d.f. of Gamma Distribution $\Gamma(\frac{1}{2}, \frac{1}{2})$, i.e. we proved that $X^2 \sim \Gamma(\frac{1}{2}, \frac{1}{2})$. Using Lemma above proves that $X_1^2 + \ldots + X_n^2 \sim \Gamma(\frac{n}{2}, \frac{1}{2})$.

□

Fisher F-distribution. Let us consider two independent random variables,

$$X \sim \chi^2_k = \Gamma\Big(\frac{k}{2}, \frac{1}{2}\Big) \quad \text{and} \quad Y \sim \chi^2_m = \Gamma\Big(\frac{m}{2}, \frac{1}{2}\Big).$$

Definition: *Distribution of the random variable*

$$Z = \frac{X/k}{Y/m}$$

is called a Fisher distribution *with degrees of freedom k and m, is denoted by $F_{k,m}$.*

First of all, let us notice that since $X \sim \chi^2_k$ can be represented as $X_1^2 + \ldots + X_k^2$ for i.i.d. standard normal $X_1, \ldots, X_k$, by law of large numbers,

$$\frac{1}{k}(X_1^2 + \ldots + X_k^2) \to \mathbb{E}X_1^2 = 1$$

when $k \to \infty$. This means that when k is large, the numerator X/k will 'concentrate' near 1. Similarly, when m gets large, the denominator Y/m will concentrate near 1. This means that when both k and m get large, the distribution $F_{k,m}$ will concentrate near 1.

Another property that is sometimes useful when using the tables of F-distribution is that

$$F_{k,m}(c, \infty) = F_{m,k}\Big(0, \frac{1}{c}\Big).$$

This is because

$$F_{k,m}(c, \infty) = \mathbb{P}\Big(\frac{X/k}{Y/m} \geq c\Big) = \mathbb{P}\Big(\frac{Y/m}{X/k} \leq \frac{1}{c}\Big) = F_{m,k}\Big(0, \frac{1}{c}\Big).$$

Next we will compute the p.d.f. of $Z \sim F_{k,m}$. Let us first compute the p.d.f. of

$$\frac{k}{m}Z = \frac{X}{Y}.$$

The p.d.f. of X and Y are

$$f(x) = \frac{(\frac{1}{2})^{\frac{k}{2}}}{\Gamma(\frac{k}{2})} x^{\frac{k}{2}-1} e^{-\frac{1}{2}x} \quad \text{and} \quad g(y) = \frac{(\frac{1}{2})^{\frac{m}{2}}}{\Gamma(\frac{m}{2})} y^{\frac{m}{2}-1} e^{-\frac{1}{2}y}$$

correspondingly, where $x \geq 0$ and $y \geq 0$. To find the p.d.f of the ratio X/Y, let us first write its c.d.f. Since X and Y are always positive, their ratio is also positive and, therefore, for $t \geq 0$ we can write:

$$\mathbb{P}\Big(\frac{X}{Y} \leq t\Big) = \mathbb{P}(X \leq tY) = \int_0^\infty \Big(\int_0^{ty} f(x)g(y)dx\Big)dy$$

since $f(x)g(y)$ is the joint density of X, Y. Since we integrate over the set $\{x \leq ty\}$ the limits of integration for x vary from 0 to ty.

Since p.d.f. is the derivative of c.d.f., the p.d.f. of the ratio X/Y can be computed as follows:

$$\begin{aligned}
\frac{d}{dt}\mathbb{P}\Big(\frac{X}{Y} \leq t\Big) &= \frac{d}{dt}\int_0^\infty \int_0^{ty} f(x)g(y)dxdy = \int_0^\infty f(ty)g(y)ydy \\
&= \int_0^\infty \frac{(\frac{1}{2})^{\frac{k}{2}}}{\Gamma(\frac{k}{2})}(ty)^{\frac{k}{2}-1}e^{-\frac{1}{2}ty}\frac{(\frac{1}{2})^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}y^{\frac{m}{2}-1}e^{-\frac{1}{2}y}ydy \\
&= \frac{(\frac{1}{2})^{\frac{k+m}{2}}}{\Gamma(\frac{k}{2})\Gamma(\frac{m}{2})}t^{\frac{k}{2}-1}\underbrace{\int_0^\infty y^{(\frac{k+m}{2})-1}e^{-\frac{1}{2}(t+1)y}dy}
\end{aligned}$$

The function in the underbraced integral almost looks like a p.d.f. of gamma distribution $\Gamma(\alpha,\beta)$ with parameters $\alpha=(k+m)/2$ and $\beta=1/2$, only the constant in front is missing. If we miltiply and divide by this constant, we will get that,

$$\begin{aligned}\frac{d}{dt}\mathbb{P}\Big(\frac{X}{Y}\le t\Big) &= \frac{(\frac{1}{2})^{\frac{k+m}{2}}}{\Gamma(\frac{k}{2})\Gamma(\frac{m}{2})}t^{\frac{k}{2}-1}\frac{\Gamma(\frac{k+m}{2})}{(\frac{1}{2}(t+1))^{\frac{k+m}{2}}}\int_0^\infty \frac{(\frac{1}{2}(t+1))^{\frac{k+m}{2}}}{\Gamma(\frac{k+m}{2})}y^{(\frac{k+m}{2})-1}e^{-\frac{1}{2}(t+1)y}dy \\ &= \frac{\Gamma(\frac{k+m}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{m}{2})}t^{\frac{k}{2}-1}(1+t)^{\frac{k+m}{2}},\end{aligned}$$

since the p.d.f. integrates to 1. To summarize, we proved that the p.d.f. of $(k/m)Z=X/Y$ is given by

$$f_{X/Y}(t)=\frac{\Gamma(\frac{k+m}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{m}{2})}t^{\frac{k}{2}-1}(1+t)^{-\frac{k+m}{2}}.$$

Since

$$\mathbb{P}(Z\le t)=\mathbb{P}\Big(\frac{X}{Y}\le\frac{kt}{m}\Big)\implies f_Z(t)=\frac{\partial}{\partial t}\mathbb{P}(Z\le t)=f_{X/Y}\Big(\frac{kt}{m}\Big)\frac{k}{m},$$

this proves that the p.d.f. of $F_{k,m}$-distribution is

$$\begin{aligned}f_{k,m}(t) &= \frac{\Gamma(\frac{k+m}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{m}{2})}\frac{k}{m}\Big(\frac{kt}{m}\Big)^{\frac{k}{2}-1}\Big(1+\frac{kt}{m}\Big)^{-\frac{k+m}{2}}. \\ &= \frac{\Gamma(\frac{k+m}{2})}{\Gamma(\frac{k}{2})\Gamma(\frac{m}{2})}k^{k/2}m^{m/2}t^{\frac{k}{2}-1}(m+kt)^{-\frac{k+m}{2}}.\end{aligned}$$

Student t_n-distribution. Let us recall that we defined t_n-distibution as the distribution of a random variable

$$T=\frac{X_1}{\sqrt{\frac{1}{n}(Y_1^2+\cdots+Y_n^2)}}$$

if $X_1,Y_1,\ldots,Y_n$ are i.i.d. standard normal. Let us compute the p.d.f. of T. First, we can write,

$$\mathbb{P}(-t\le T\le t)=\mathbb{P}(T^2\le t^2)=\mathbb{P}\Big(\frac{X_1^2}{(Y_1^2+\cdots+Y_n^2)/n}\le t^2\Big).$$

If $f_T(x)$ denotes the p.d.f. of T then the left hand side can be written as

$$\mathbb{P}(-t\le T\le t)=\int_{-t}^t f_T(x)dx.$$

On the other hand, by definition,

$$\frac{X_1^2}{(Y_1^2+\ldots+Y_n^2)/n}$$

has Fisher $F_{1,n}$-distribution and, therefore, the right hand side can be written as

$$\int_0^{t^2} f_{1,n}(x)dx.$$

We get that,

$$\int_{-t}^{t} f_T(x)dx = \int_{0}^{t^2} f_{1,n}(x)dx.$$

Taking derivative of both side with respect to t gives

$$f_T(t) + f_T(-t) = f_{1,n}(t^2)2t.$$

But $f_T(t) = f_T(-t)$ since the distribution of T is obviously symmetric, because the numerator X has symmetric distribution $N(0,1)$. This, finally, proves that

$$f_T(t) = f_{1,n}(t^2)t = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2})}\frac{1}{\sqrt{n}}(1+\frac{t^2}{n})^{-\frac{n+1}{2}}.$$

□