Parameter Estimation
Fitting Probability Distributions
Method of Moments

MIT 18.443

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Outline

Method of Moments

One-Sample Model

- $X_1, X_2, \ldots, X_n$ i.i.d. r.v.'s. with density function $f(x \mid \theta)$
  Joint density of $X = (X_1, X_2, \ldots, X_n)$ is given by:

$$f(x_1, \ldots, x_n \mid \theta) \quad = \quad f(x_1 \mid \theta) \times \cdots \times f(x_n \mid \theta) = \prod_{i=1}^{n} f(x_i \mid \theta)$$

Population and Sample Moments

- $\mu_k = E[X^k]$: kth population moment, where k is a positive integer.
- $\hat{\mu}_k = \frac{1}{n} \sum_{i=1}^{n} x_i^k$: kth sample moment.

Method of Moments

Method of Moments

1. Calculate low-order moments, as functions of θ
2. Set up a system of equations setting the population moments (as functions of the parameters in step 1) equal to the sample moments, and derive expressions for the parameters as functions of the sample moments.
3. Insert the sample moments into the solutions of step 2.

Outline

Example: Gamma Distribution

Example 8.4.A Poisson Distribution

For $X1, X_2, \ldots, X_n$ i.i.d. $Poisson(\lambda)$,

- $\lambda = E[X]$
- $\hat{\mu}_1 = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$
- Method of Moments Estimate: $\hat{\lambda}_{MOM} = \overline{X}$.
- Note: $Var(X) = E[X^2] - (E[X])^2 = \mu_2 - [\mu_1]^2 = \lambda$.
  So, an alternative is $\hat{\lambda}^*_{MOM} = \hat{\mu}_2 - \hat{\mu}_1^2$.

Method of Moments: Poisson Distribution

Method-of-Moments Estimate: $\hat{\lambda}_{MOM} = \overline{X}$.

- The **sampling distribution** of $\hat{\lambda}_{MOM}$ is well-defined.
- $n\hat{\lambda}_{MOM} = S = \sum_{i=1}^{n} X_i$ where X_i iid $Poisson(\lambda)$.
- $S \sim Poisson(n\lambda)$.
- Because $E[S] = n\lambda$ and $Var[S] = n\lambda$, it follows that

$$E[\hat{\lambda}_{MOM}] = \lambda \text{ and } Var[\hat{\lambda}_{MOM}] = \lambda/n.$$

- $\hat{\lambda}_{MOM}$ is **unbiased**.
- As $n \to \infty$, $\hat{\lambda}_{MOM} \to \lambda$ and the sampling distribution concentrates about λ, with the standard deviation:

$$\sigma_{\hat{\lambda}_{MOM}} = \sqrt{\frac{\lambda}{n}} \text{ (the \textbf{standard error} of } \hat{\lambda})$$

- For large n, by the Central Limit Theorem (CLT),

$$[\frac{\hat{\lambda}_{MOM} - \lambda}{\sigma_{\hat{\lambda}_{MOM}}}] \xrightarrow{\mathcal{L}} N(0,1).$$

Method of Moments: Normal Distribution

Example 8.4.B Normal Distribution

- For $X1, X_2, \ldots, X_n$ i.i.d. $Normal(\mu, \sigma^2)$,

$$\begin{array}{rcccl} \mu_1 & = & E(X) & = & \mu \\ \mu_2 & = & E(X^2) & = & \mu^2 + \sigma^2 \end{array}$$

- Solve for $\theta = (\mu, \sigma^2)$

$$\begin{array}{rcl} \mu & = & \mu_1 \\ \sigma^2 & = & \mu_2 - \mu_1^2 \end{array}$$

- $\hat{\mu} = \hat{\mu}_1 = \overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$
- $\hat{\sigma}^2 = \hat{\mu}_2 - \hat{\mu}_1^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \overline{X}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{X})^2$
- Sampling distributions:

$$\begin{array}{rcl} \hat{\mu} & \sim & N(\mu, \sigma^2/n) \\ (\frac{n}{\sigma^2})\hat{\sigma}^2 & \sim & \chi^2_{n-1} \text{ (independent of } \hat{\mu}) \end{array}$$

Method of Moments: Gamma Distribution

Gamma Distribution as Sum of IID Random Variables

- Define $W_1, \ldots, W_K$ i.i.d. $Gamma(\alpha, \lambda)$ random variables
- Density function:

$$f(w \mid \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} w^{\alpha-1} e^{-\lambda w}, w > 0$$

- Moment-generating function:

$$M_{W_j}(t) = E[e^{tW_j}] = (1 - \frac{t}{\lambda})^{-\alpha}$$

- Moment-generating function of $V = W_1 + W_2 + \cdots W_k$:

$$\begin{aligned} M_V(t) &= E[e^{tV}] = E[e^{t(W_1 + \cdots W_k)}] \\ &= E[e^{tW_1}]E[e^{tW_2}] \cdots E[e^{tW_k}] \\ &= (1 - \frac{t}{\lambda})^{-K\alpha} \end{aligned}$$

So $\quad V \sim Gamma(k\alpha, \lambda)$.

Method of Moments: Gamma Distribution

Gamma Distribution as Sum of IID Random Variables

- The Gamma distribution models the total waiting time for k successive events where each event has a waiting time of $Gamma(\alpha/k, \lambda)$.
- $Gamma(1, \lambda)$ is an $Exponential(\lambda)$ distribution
- $Gamma(k, \lambda)$ is distribution of sum of K iid $Exponential(\lambda)$ r.v.s

Method of Moments: Gamma Distribution

Moments of Gamma Distribution

- $W \sim Gamma(\alpha, \lambda)$ with mgf $M_W(t) = E[e^{tW}] = (1 - \frac{t}{\lambda})^{-\alpha}$
- $\mu_1 = E[W] = M'_W(t = 0) = \dfrac{\alpha}{\lambda}$
- $\mu_2 = E[W^2] = M''_W(t = 0) = \dfrac{\alpha(\alpha + 1)}{\lambda^2}$
- $\mu_2 - \mu_1^2 = Var[W] = M''_W(t = 0) - (E[W])^2 = \dfrac{\alpha}{\lambda^2}$

Method-of-Moments(MOM) Estimator

- $\hat{\lambda}_{MOM} = \dfrac{\hat{\mu}_1}{\hat{\mu}_2 - \hat{\mu}_1^2} = \dfrac{\overline{W}}{\hat{\sigma}_W^2}$
- $\hat{\alpha}_{MOM} = \hat{\lambda}\mu_1 = \dfrac{\hat{\mu}_1^2}{\hat{\mu}_2 - \hat{\mu}_1^2} = \dfrac{\overline{W}^2}{\hat{\sigma}_W^2}$

Method of Moments: Gamma Distribution

Method-of-Moments(MOM) Estimator

- $\hat{\lambda}_{MOM} = \dfrac{\hat{\mu}_1}{\hat{\mu}_2 - \hat{\mu}_1^2} = \dfrac{\overline{W}}{\hat{\sigma}_W^2}$
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NOTE:

- MOM Estimator of λ is ratio of sample mean to variance (units=?)
- MOM Estimator of α is ratio of squared sample mean to variance (units=?)
- λ is the **rate** parameter and α is the **shape parameter**.
- What are the sampling distributions of $\hat{\lambda}$ and $\hat{\alpha}$?

Method-of-Moments: Consistency

Parameter Estimation Context

- Suppose $X_1, \ldots, X_n$ are iid with density/pmf function $f(x \mid \theta)$.
- Let $\hat{\theta}_n = \hat{\theta}(\mathbf{X}_n)$ be an estimate of θ based on $\mathbf{X}_n = (X_1, \ldots, X_n)$

Definition The estimate $\hat{\theta}_n$ is **consistent** for θ if for any $\epsilon > 0$,

$$\lim_{n \to \infty} P(|\hat{\theta}_n - \theta| > \epsilon) = 0.$$

Consistency of Method-of-Moments Estimates

- If μ_k is finite, then $\hat{\mu}_k$ is consistent for μ_k.
- Suppose the method-of-moments equations provide a one-to-one estimate of θ given the first k^* sample moments. Then, if the first k^* population moments exist, the method-of-moments estimate is consistent.

Method-of-Moments: Sampling Distributions

Sampling Distribution of Method-of-Moments Estimates

- For special cases, the sampling distribution of $\hat{\theta}_{MOM}$ is exactly known by probability theory

  E.g., Normal, Binomial, Poisson, Exponential
- In general, **Bootstrap** (Monte Carlo simulation) methods provide approximations to the sampling distributions of MOM estimates.

  E.g., $Gamma(\alpha, \lambda)$ distribution, unknown α (shape)
- Limiting distributions can be derived
  - Apply Central Limit Theorem to obtain limiting distribution of sample moments.
  - Apply transformation of variables to obtain limiting distribution of MOM estimates.

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