18.445 HOMEWORK 3 SOLUTIONS

Exercise 9.2. An Oregon professor has n umbrellas, of which initially $k \in (0, n)$ are at his office and $n - k$ are at his home. Every day, the professor walks to the office in the morning and returns home in the evening. In each trip, he takes and umbrella with him only if it is raining. Assume that in every trip between home and office or back, the chance of rain is $p \in (0, 1)$, independently of other trips.

(a). Asymptotically, in what fraction of his trips does the professor get wet?

Proof. First, we identify a Markov chain with $2n + 2$ states. For $i \in [0, n]$, let x_{2i+1} denote the state that the professor is at home and there are i umbrellas at home and $n - i$ umbrellas at the office; let x_{2i+2} denote the state that the professor is at the office and there are i umbrellas at home and $n - i$ umbrellas at the office. Then the transition matrix of the Markov chain is defined by

$$
\begin{aligned}
&P_{j,k} = 0 \text{ if } j = k \text{ or } |j - k| > 1 \text{ for } j, k \in [1, 2n + 2],\\
&P_{1,2} = P_{2n+2,2n+1} = 1,\\
&P_{2i+1,2i} = P_{2i,2i+1} = p \text{ for } i \in [1, n],\\
&P_{2i+1,2i+2} = P_{2i+2,2i+1} = 1 - p \text{ for } i \in [0, n].
\end{aligned}
$$

The following diagram gives an intuition of the chain:

$$x_1 \underset{1-p}{\overset{1}{\rightleftharpoons}} x_2 \underset{p}{\overset{p}{\rightleftharpoons}} x_3 \underset{1-p}{\overset{1-p}{\rightleftharpoons}} x_4 \underset{p}{\overset{p}{\rightleftharpoons}} \cdots\cdots \underset{1-p}{\overset{1-p}{\rightleftharpoons}} x_{2n} \underset{p}{\overset{p}{\rightleftharpoons}} x_{2n+1} \underset{1}{\overset{1-p}{\rightleftharpoons}} x_{2n+2}.$$

To find a stationary distribution π, we need $\pi P = \pi$. It is easy to observe that $\pi(x_1) = \pi(x_{2n+2}) = a$ and $\pi(x_j) = b$ for $j \in [2, 2n + 1]$, where a and b satisfy $a = b(1 - p)$ and $2a + 2nb = 1$. Hence

$$\pi(x_1) = \pi(x_{2n+2}) = \frac{1-p}{2-2p+2n} \quad \text{and} \quad \pi(x_j) = \frac{1}{2-2p+2n} \text{ for } j \in [2, 2n + 1].$$

Since the professor gets wet when it is raining and the chain moves from x_1 to x_2 or from x_{2n+2} to x_{2n+1}, we conclude that asymptotically this probability is

$$p\pi(x_1) + p\pi(x_{2n+2}) = \frac{2p(1-p)}{2-2p+2n}.$$

□

(b). Determine the expected number of trips until all n umbrellas are at the same location.

Proof. Note that all n umbrellas are at the same location if and only if the chain is at x_1, x_2, x_{2n+1} or x_{2n+2}. Since the chain can only move from x_i to x_{i-1} or x_{i+1} in one step, if it starts at x_j where $j \in [3, 2n]$, we need to compute the expectation of the hitting time τ_j of $\{x_2, x_{2n+1}\}$.

Let $a_j = \mathbb{E}[\tau_j]$. Then $a_2 = a_{2n+1} = 0$, and by symmetry $a_{n+1} = a_{n+2}$. It is easily seen that for $i \in [3, n+1]$,

$$a_i = 1 + pa_{i-1} + (1 - p)a_{i+1} \text{ if } i \text{ is odd,} \tag{1}$$

$$a_i = 1 + (1 - p)a_{i-1} + pa_{i+1} \text{ if } i \text{ is even.} \tag{2}$$

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If n is odd, since $a_{n+1} = a_{n+2}$, then (2) applied to $n+1$ gives $(1-p)a_{n+1} = 1 + (1-p)a_n$. Thus (1) applied to n gives $pa_n = 2 + pa_{n-1}$. Proceeding backward inductively, we see that

$$\begin{aligned}
(1-p)a_{n+1} &= 1 + (1-p)a_n, \\
pa_n &= 2 + pa_{n-1}, \\
(1-p)a_{n-1} &= 3 + (1-p)a_{n-2}, \\
pa_{n-2} &= 4 + pa_{n-3}, \\
&\vdots \\
(1-p)a_4 &= n-2 + (1-p)a_3, \\
pa_3 &= n-1 + pa_2.
\end{aligned}$$

On the other hand, if n is odd, then similarly we have

$$\begin{aligned}
pa_{n+1} &= 1 + pa_n, \\
(1-p)a_n &= 2 + (1-p)a_{n-1}, \\
&\vdots \\
(1-p)a_4 &= n-2 + (1-p)a_3, \\
pa_3 &= n-1 + pa_2.
\end{aligned}$$

Since $a_2 = 0$, $a_3 = \frac{n-1}{p}$. Inductively we can solve for all a_i for $i \in [2, 2n+2]$ using the above relation and symmetry. In particular, if the professor is at home with $n-k$ umbrellas at home and k umbrellas at the office, then the chain starts at $x_{2n-2k+1}$. If $k \geq n/2$, then using the above relation,

$$\mathbb{E}[\tau_{2n-2k+1}] = a_{2n-2k+1} = \frac{n-1}{p} + \frac{n-2}{1-p} + \frac{n-3}{p} + \cdots + \frac{2k-n+1}{p} = \frac{k(n-k)}{p} + \frac{k(n-k-1)}{1-p}.$$

If $k < n/2$, then by symmetry $a_{2n-2k+1} = a_{2k+2}$, so

$$\mathbb{E}[\tau_{2n-2k+1}] = a_{2k+2} = \frac{n-1}{p} + \frac{n-2}{1-p} + \frac{n-3}{p} + \cdots + \frac{n-2k}{1-p} = \frac{k(n-k)}{p} + \frac{k(k-n-1)}{1-p}$$

which is the same formula. We conclude that the expected number of trips until all n umbrellas are at the same location is

$$\frac{k(n-k)}{p} + \frac{k(k-n-1)}{1-p}.$$

$\square$

(c). Determine the expected number of trips until the professor gets wet.

Proof. Denote the expectation from Part (b) by $c = \frac{k(n-k)}{p} + \frac{k(k-n-1)}{1-p}$. The professor gets wet if it is raining and the chain moves from x_1 to x_2 or from x_{2n+2} to x_{2n+1}. This happens only after the chain gets to x_2 or x_{2n+1}, so we can compute the expectation assuming that the chain starts at x_2 or x_{2n+1} and then add c.

Let a be the expected number of trips until the professor gets wet when the chain starts at x_2. Let b be the expected number when the chain starts at x_1. Recall that $a_3 = \frac{n-1}{p}$ from Part (b). Then one-step analysis starting from x_2 and x_1 gives $a = 1 + (1-p)b + p(a_3 + a)$ and $b = 1 + (1-p)a$ respectively. Therefore $a = \frac{n+1-p}{p(1-p)}$. We conclude that the expected total number of trips until the professor gets wet is

$$\frac{k(n-k)}{p} + \frac{k(k-n-1)}{1-p} + \frac{n+1-p}{p(1-p)}.$$

$\square$

Exercise 9.4. Let θ be a flow from a to z which satisfies both the cycle law and $\|\theta\| = \|I\|$. Define a function h on nodes by

$$h(x) = \sum_{i=1}^{m} [\theta(\vec{e}_i) - I(\vec{e}_i)] r(\vec{e}_i),$$

where $\vec{e}_1, \ldots, \vec{e}_m$ is an arbitrary path from a to x.

(a). Show that h is well-defined and harmonic at all nodes.

Proof. Let $\vec{f}_1, \ldots, \vec{f}_n$ be another path from a to x. Then it differs from $\vec{e}_1, \ldots, \vec{e}_m$ by cycles in the sense that for some $1 \le i \le j \le m$ and $1 \le k \le l \le n$, $\vec{e}_i, \vec{e}_{i+1}, \ldots, \vec{e}_j, -\vec{f}_l, -\vec{f}_{l-1}, \ldots, -\vec{f}_k$ form a cycle (there may be more than one cycles). By the cycle law, the function $[\theta(\cdot) - I(\cdot)]r(\cdot)$ sums to zero over such a cycle. Therefore if we replace $\vec{e}_i, \vec{e}_{i+1}, \ldots, \vec{e}_j$ by $\vec{f}_k, \vec{f}_{k+1}, \ldots, \vec{f}_l$, the value of h does not change. Inductively replacing all cycles, we see that h takes the same value for all arbitrary paths from a to x. Thus h is well-defined.

Moreover, for $y \sim x$, let $\vec{e}_y$ denote the edge from x to y. We can write $h(y) = h(x) + [\theta(\vec{e}_y) - I(\vec{e}_y)]r(\vec{e}_y)$. Therefore,

$$\begin{aligned}
\sum_{y \sim x} P(x,y)h(y) &= \sum_{y \sim x} \frac{c(x,y)}{c(x)} h(y) \\
&= \sum_{y \sim x} \frac{c(x,y)}{c(x)} \big(h(x) + [\theta(\vec{e}_y) - I(\vec{e}_y)]r(\vec{e}_y)\big) \\
&= h(x) + \frac{1}{c(x)} \sum_{y \sim x} [\theta(\vec{e}_y) - I(\vec{e}_y)] \\
&= h(x) + \frac{1}{c(x)} [\operatorname{div} \theta(x) - \operatorname{div} I(x)] \\
&= h(x)
\end{aligned}$$

for $x \notin \{a, z\}$ by the node law and for $x = a$ by $\|\theta\| = \|I\|$. The formula also holds for $x = z$ because $\sum_{x \in V} \operatorname{div} \theta(x) = 0$ for any flow so that $\operatorname{div} \theta(z) - \operatorname{div} I(z) = 0$. Thus h is harmonic at all nodes. □

(b). Use Part (a) to give an alternative proof of Proposition 9.4.

Proof. Trivially $h(a) = 0$, so $h \equiv 0$ is the unique harmonic extension. We conclude that $\theta = I$. □

Exercise 9.5. Show that if, in a network with source a and sink z, vertices with different voltages are glued together, then the effective resistance from a to z will strictly decrease.

Proof. By gluing vertices with different voltages, we change the old voltage W_1 to a different new voltage W_2 on the network. Let I_1 and I_2 be the unit current flows corresponding to W_1 and W_2 respectively. Then I_1 and I_2 are necessarily different. By Thomson's Principle, the old effective resistance is $\mathcal{E}(I_1)$ and the new effective resistance is $\mathcal{E}(I_2) = \inf_{\{\text{unit flow } \theta\}} \mathcal{E}(\theta)$ where I_2 is the unique minimizer. However, I_1 is still a unit flow in the network, so the effective resistance decreases strictly. □

Exercise 9.6. Show that $\mathcal{R}(a \leftrightarrow z)$ is a concave function of $\{r(e)\}$.

Proof. Consider two sets of resistors $\{r(e)\}$ and $\{r'(e)\}$. Let $\mathcal{R}(a \leftrightarrow z)$ and $\mathcal{R}'(a \leftrightarrow z)$ denote their effective resistance respectively. For $s \in [0, 1]$, define $\mathcal{R}_s(a \leftrightarrow z)$ to be the effective resistance of $\{sr(e) + (1-s)r'(e)\}$ (the resistance of e is $sr(e) + (1-s)r'(e)$). Let θ range over arbitrary unit flows from a to z. By Thomson's

Principle,

$$\begin{aligned}
\mathcal{R}_s(a \leftrightarrow z) &= \inf_\theta \mathcal{E}_s(\theta) \\
&= \inf_\theta \sum_e \theta(e)^2[sr(e) + (1-s)r'(e)] \\
&\geq s \inf_\theta \sum_e \theta(e)^2 r(e) + (1-s) \inf_\theta \sum_e \theta(e)^2 r'(e) \\
&= s \inf_\theta \mathcal{E}(\theta) + (1-s) \inf_\theta \mathcal{E}'(\theta) \\
&= s\mathcal{R}(a \leftrightarrow z) + (1-s)\mathcal{R}'(a \leftrightarrow z).
\end{aligned}$$

This proves that $\mathcal{R}(a \leftrightarrow z)$ is a concave function of $\{r(e)\}$. □

Exercise 10.1. Prove Lemma 10.5 by copying the proof in Proposition 1.14 that $\tilde{\pi}$ as defined in (1.19) satisfies $\tilde{\pi} = \tilde{\pi}P$, substituting $G_\tau(a, x)$ in place of $\tilde{\pi}(x)$.

Proof. Since τ is a stopping time, $\tau > t$ is determined by $X_0, \ldots, X_t$ and thus independent of X_{t+1}. Hence

$$\begin{aligned}
\sum_{x \in \Omega} G_\tau(a, x)P(x, y) &= \sum_x \sum_{t=0}^{\infty} \mathbb{P}_a[X_t = x, \tau > t]P(x, y) \\
&= \sum_{t=0}^{\infty} \sum_x \mathbb{P}_a[X_t = x, X_{t+1} = y, \tau > t] \\
&= \sum_{t=1}^{\infty} \mathbb{P}_a[X_t = y, \tau > t-1] \\
&= \sum_{t=0}^{\infty} \mathbb{P}_a[X_t = y, \tau > t] - \mathbb{P}_a[X_0 = y, \tau > 0] + \sum_{t=1}^{\infty} \mathbb{P}_a[X_t = y, \tau = t] \\
&= G_\tau(a, y) - \mathbb{P}_a[X_0 = y] + \mathbb{P}_a[X_\tau = y] \\
&= G_\tau(a, y)
\end{aligned}$$

where the last equality holds because if $y = a$ then the last two terms are both 1 and if $y \neq a$ the last two terms are both 0. This establishes the stationarity. Since

$$\sum_x G_\tau(a, x) = \sum_x \sum_{t=0}^{\infty} \mathbb{P}_a[X_t = x, \tau > t] = \sum_{t=0}^{\infty} \mathbb{P}_a[\tau > t] = \mathbb{E}_a[\tau],$$

we normalize $G(a, x)$ to get the stationary distribution

$$\pi(x) = \frac{G_\tau(a, x)}{\mathbb{E}_a(\tau)}.$$

□

Exercise 10.3. Let G be a connected graph on at least 3 vertices in which the vertex v has only one neighbor, namely w. Show that for the simple random walk on G, $\mathbb{E}_v\tau_w \neq \mathbb{E}_w\tau_v$.

Proof. Since G has at least 3 vertices and v only has one neighbor, w must have a neighbor u different from v. We have $\mathbb{E}_v\tau_w = 1$ and

$$\mathbb{E}_w\tau_v \geq 1 + \frac{\mathbb{E}_u\tau_v}{\deg(w)} \geq 1 + \frac{2}{\deg(w)} > 1,$$

so they are not equal. □

Exercise 10.4. Consider simple random walk on the binary tree of depth k with $n = 2^{k+1} - 1$ vertices (first defined in Section 5.3.4).

(a). Let a and b be two vertices at level m whose most recent common ancestor c is at level $h < m$. Show that $\mathbb{E}_a\tau_b = \mathbb{E}_a\tau_{a,c}$ and find its value.

Proof. Since the random walk starting from a must visit c before visiting b, we have $\mathbb{E}_a\tau_b = \mathbb{E}_a\tau_c + \mathbb{E}_c\tau_b$. On the other hand, $\mathbb{E}_a\tau_{a,c} = \mathbb{E}_a\tau_c + \mathbb{E}_c\tau_a$ by the Commute Time Identity. Moreover, $\mathbb{E}_c\tau_a = \mathbb{E}_c\tau_b$ by symmetry. We conclude that $\mathbb{E}_a\tau_b = \mathbb{E}_a\tau_{a,c}$.

If we assume that the tree has unit resistance on each edge, then $c_G = 2+3(2^k-2)+2^k = 2^{k+2}-4 = 2n-2$. By Example 9.7, $\mathcal{R}(a \leftrightarrow c)$ is the length of the path joining a and c, i.e. $m-h$ in this case. Hence the Commuter Time Identity implies that

$$\mathbb{E}_a\tau_b = \mathbb{E}_a\tau_{a,c} = c_G\mathcal{R}(a \leftrightarrow c) = (2n-2)(m-h).$$

□

(b). Show that the maximal value of $\mathbb{E}_a\tau_b$ is achieved when a and b are leaves whose most recent common ancestor is the root of the tree.

Proof. First, if a and b are at different levels, we may assume without loss of generality that a is at level m and b is at level h where $m > h$. Let d be a descendant of b at level m. Then any random walk starting from a must visit b before visiting d, so $\mathbb{E}_a\tau_b \leq \mathbb{E}_a\tau_d$. Hence to achieve the maximal value of $\mathbb{E}_a\tau_b$, we may assume that a and b are at the same level. In this case, the result of Part (a) implies that the maximum $(2n-2)m$ is achieved when the most common ancestor c is the root of the tree. □

Exercise 21.1. Use the Strong Law of Large Numbers to give a proof that the biased random walk in Example 21.2 is transient.

Proof. Suppose the chain starts at $X_0 = x$ and $X_t = x + \sum_{s=1}^t Y_s$ where Y_s are i.i.d. and $Y_s = -1$ with probability q and $Y_s = 1$ with probability p. The Strong Law of Large Numbers implies that a.s.

$$\lim_{t\to\infty} \frac{X_t}{t} = \mathbb{E}[Y_s] = p - q > 0.$$

Hence a.s. $X_t > t(p-q)/2$ for t sufficiently large, so a.s. the random walk only visits any fixed state y finitely many times. Since the number of visits to y is a geometric variable with parameter $\mathbb{P}_x[\tau_y = \infty]$, this quantity is positive. Hence Proposition 21.3 implies that $\mathbb{P}_x[\tau_x^+ < \infty] < 1$, i.e. the chain is transient. □

Exercise 21.2. Suppose that P is irreducible. Show that if $\pi = \pi P$ for a probability distribution π, then $\pi(x) > 0$ for all $x \in \Omega$.

Proof. Suppose $\pi(x) = 0$ for some state $x \in \Omega$. Then

$$0 = \pi(x) = \sum_{y\in\Omega} \pi(y)P(y,x), \tag{3}$$

so each term on the right-hand side is 0. Since the chain is irreducible, for each y there exists a sequence $z_0 = y, z_1, \ldots, z_n = x$ such that $P(z_{i-1}, z_i) > 0$ for $i \in [1,n]$. Thus $\pi(z_{n-1}) = 0$ by (3). Replacing x with z_{n-1} in (3), we see that $\pi(z_{n-2}) = 0$. Inductively, $\pi(z_i) = 0$ for all $i \in [0,n]$ and in particular $\pi(y) = 0$. This is absurd as y is arbitrary, so $\pi(x) > 0$ for all $x \in \Omega$. □

Exercise 21.5. Let P be an irreducible and aperiodic transition matrix on Ω. Let $\tilde{P}$ be the matrix on $\Omega\times\Omega$ defined by

$$\tilde{P}((x,y),(z,w)) = P(x,z)P(y,w), \quad (x,y) \in \Omega\times\Omega, (z,w) \in \Omega\times\Omega.$$

Show that $\tilde{P}$ is irreducible.

Proof. Let $A_{xy} = \{t : P^t(x,y) > 0\}$. Since P is irreducible, A_{xy} is not empty. Since P is aperiodic, $\gcd(A_{xx}) = 1$. Because A_{xx} is closed under addition, there exists t_x such that $t \in A_{xx}$ for $t \geq t_x$.

Choose s so that $P^s(x,y) > 0$. Then $P^{t+s}(x,y) \geq P^t(x,x)P^s(x,y) > 0$ for $t \geq t_x$. Hence $t \in A_{xy}$ for all $t \geq t_{xy} := t_x + s$. Therefore, for $t \geq t_{xz} \vee t_{yw}$,

$$\tilde{P}^t((x,y),(z,w)) = P^t(x,z)P^t(y,w) > 0,$$

so $\tilde{P}$ is irreducible. □

Exercise 21.8. Let P be the transition matrix for simple random walk on $\mathbb{Z}$. Show that the walk is not positive recurrent by showing there are no probability distributions π on $\mathbb{Z}$ satisfying $\pi P = \pi$.

Proof. Theorem 21.12 gives the equivalence of positive recurrence and the existence of a stationary distribution, so it suffices to show that there does not exist a stationary distribution.

Suppose there exists π such that $\pi = \pi P$. For any $n \in \mathbb{Z}$,

$$\pi(n) = \frac{1}{2}\pi(n-1) + \frac{1}{2}\pi(n+1),$$

so $\pi(n) - \pi(n-1) = \pi(n+1) - \pi(n)$. If this difference is zero, then $\pi(n)$ is constant, which cannot be true as there are infinitely many states; if the difference is not zero, then π is not bounded, which is again a contradiction. $\square$