18.445 Homework 4, Due April 22th, 2015

Exercise 1. Let X, Y be two random variables on $(\Omega, \mathscr{F}, \mathbb{P})$. Let $\mathscr{A} \subset \mathscr{F}$ be a sub-σ-algebra. The random variables X and Y are said to be independent conditionally on $\mathscr{A}$ is for every non-negative measurable functions f, g, we have

$$\mathbb{E}[f(X)g(Y) \,|\, \mathscr{A}] = \mathbb{E}[f(X) \,|\, \mathscr{A}] \times \mathbb{E}[g(Y) \,|\, \mathscr{A}] \quad a.s.$$

Show that X, Y are independent conditionally on $\mathscr{A}$ if and only if for every non-negative $\mathscr{A}$-measurable random variable Z, and every non-negative measurable functions f, g, we have

$$\mathbb{E}[f(X)g(Y)Z] = \mathbb{E}[f(X)Z\mathbb{E}[g(Y) \,|\, \mathscr{A}]].$$

Exercise 2. Let $X = (X_n)_{n \geq 0}$ be a martingale.

(1) Suppose that T is a stopping time, show that X^T is also a martingale. In particular, $\mathbb{E}[X_{T \wedge n}] = \mathbb{E}[X_0]$.

(2) Suppose that $S \leq T$ are bounded stopping times, show that $\mathbb{E}[X_T \,|\, \mathscr{F}_S] = X_S, a.s.$ In particular, $\mathbb{E}[X_T] = \mathbb{E}[X_S]$.

(3) Suppose that there exists an integrable random variable Y such that $|X_n| \leq Y$ for all n, and T is a stopping time which is finite a.s., show that $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

(4) Suppose that X has bounded increments, i.e. $\exists M > 0$ such that $|X_{n+1} - X_n| \leq M$ for all n, and T is a stopping time with $\mathbb{E}[T] < \infty$, show that $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Exercise 3. Let $X = (X_n)_{n \geq 0}$ be Gambler's ruin with state space $\Omega = \{0, 1, 2, ..., N\}$:

$$X_0 = k, \quad \mathbb{P}[X_{n+1} = X_n + 1 \,|\, X_n] = \mathbb{P}[X_{n+1} = X_n - 1 \,|\, X_n] = 1/2, \quad \tau = \min\{n : X_n = 0 \text{ or } N\}.$$

(1) Show that $Y = (Y_n := X_n^2 - n)_{n \geq 0}$ is a martingale.

(2) Show that Y has bounded increments.

(3) Show that $\mathbb{E}[\tau] < \infty$.

(4) Show that $\mathbb{E}[\tau] = k(N - k)$.

Exercise 4. Let $X = (X_n)_{n \geq 0}$ be the simple random walk on $\mathbb{Z}$.

(1) Show that $(Y_n := X_n^3 - 3nX_n)_{n \geq 0}$ is a martingale.

(2) Let τ be the first time that the walker hits either 0 or N. Show that, for $0 \leq k \leq N$, we have

$$\mathbb{E}_k[\tau \,|\, X_\tau = N] = \frac{N^2 - k^2}{3}.$$

Exercise 5. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with filtration $(\mathscr{F}_n)_{n\geq 0}$.

(1) For any $m, m' \geq n$ and $A \in \mathscr{F}_n$, show that $T = m1_A + m'1_{A^c}$ is a stopping time.

(2) Show that an adapted process $(X_n)_{n\geq 0}$ is a martingale if and only if it is integrable, and for every bounded stopping time T, we have $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Exercise 6. Let $X = (X_n)_{n\geq 0}$ be a martingale in L^2.

(1) Show that its increments $(X_{n+1} - X_n)_{n\geq 0}$ are pairwise orthogonal, i.e. for all $n \neq m$, we have

$$\mathbb{E}[(X_{n+1} - X_n)(X_{m+1} - X_m)] = 0.$$

(2) Show that X is bounded in L^2 if and only if

$$\sum_{n\geq 0} \mathbb{E}[(X_{n+1} - X_n)^2] < \infty.$$