(1) Prove that for $0 < \mu \le 1/2$

$$D(1-\mu+t, 1-\mu) \ge \frac{t^2}{2\mu(1-\mu)},$$

where $D(p,q)$ is a Kullback-Leibler divergence.

(2) Prove that if $(x'_1, \ldots, x'_n)$ is an independent copy of $(x_1, \ldots, x_n)$ then

$$P\Big(\sum_{i=1}^n (f(x_i) - f(x'_i)) > (2t \sum_{i=1}^n (f(x_i) - f(x'_i))^2)^{1/2}\Big) \le e^{-t}.$$

(3a) Let ξ be a Poisson random variable with parameter λ, i.e.

$$P(\xi = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \ \ k = 0, 1, 2, \ldots$$

Prove that

$$\frac{1}{\sqrt{2\pi(\lambda+t)} e^{1/(12(\lambda+t))}} e^{-\lambda\varphi(t/\lambda)} \le P(\xi \ge \lambda + t) \le e^{-\lambda\varphi(t/\lambda)},$$

where $\varphi(x) = (1+x)\ln(1+x) - x$. (Here you may assume that $\lambda + t$ is a natural number.) You can use the following version of Stirling's formula

$$\sqrt{2\pi k}\Big(\frac{k}{e}\Big)^k e^{1/(12k+1)} \le k! \le \sqrt{2\pi k}\Big(\frac{k}{e}\Big)^k e^{1/(12k)}.$$

(3b) Assume that $x_1, \ldots, x_n$ are i.i.d. with the distribution

$$P(x_i = 1) = \lambda/n, \quad P(x_i = 0) = 1 - \lambda/n.$$

State Bennett's inequality for $x_1 + \ldots + x_n$.

Remark. It is well known that $x_1 + \ldots + x_n$ and ξ are "close" in distribution if n is large. The point of (3a) and (3b) is to show that Bennett's inequality is sharp.

(4) Prove that VC dimension of the set of all circles on the plane is 3.

(5) For a fixed integer $d \ge 1$ let us consider VC classes of sets $\mathcal{C}_1, \ldots, \mathcal{C}_d$ with VC dimensions $V_1, \ldots, V_d$ correspondingly. Consider a new family of sets defined by

$$\{x \in \mathcal{X} : \sum_{i=1}^d \alpha_i I(x \in C_i) > 0\}$$

for any real numbers $\alpha_1, \ldots, \alpha_d$ and sets $C_i \in \mathcal{C}_i$. Prove that this new class of sets is VC.

(6) Consider a family $\mathcal{H}$ of functions on the real line of the following type

$$\pm I(x \geq a), \pm I(x > a), \pm I(x \leq a), \pm I(x < a)$$

for any real number a. Consider a convex hull of $\mathcal{H}$,

$$\mathcal{F} = \mathrm{conv}\mathcal{H} = \{\sum_{i=1}^{d} \alpha_i h_i(x) : d \geq 1, \alpha_i \geq 0, \sum_{i=1}^{d} \alpha_i = 1, h_i \in \mathcal{H}\}$$

and generate a family of sets $\mathcal{C}$ by

$$\mathcal{C} = \Big\{\{x : f(x) \geq 0\} : f \in \mathcal{F}\Big\}.$$

Prove that any finite union of disjoint intervals is an element in $\mathcal{C}$, i.e. for any $k \geq 1$, $\cup_{i \leq k}[a_i, b_i] \in \mathcal{C}$.