3 Representations of finite groups: basic results

Recall that a **representation** of a group G over a field k is a k-vector space V together with a group homomorphism $\rho : G \to GL(V)$. As we have explained above, a representation of a group G over k is the same thing as a representation of its group algebra $k[G]$.

In this section, we begin a systematic development of representation theory of finite groups.

3.1 Maschke's Theorem

Theorem 3.1. *(Maschke) Let G be a finite group and k a field whose characteristic does not divide $|G|$. Then:*

(i) The algebra $k[G]$ is semisimple.

(ii) There is an isomorphism of algebras $\psi : k[G] \to \oplus_i \mathrm{End} V_i$ defined by $g \mapsto \oplus_i g|_{V_i}$, where V_i are the irreducible representations of G. In particular, this is an isomorphism of representations of G (where G acts on both sides by left multiplication). Hence, the regular representation $k[G]$ decomposes into irreducibles as $\oplus_i \dim(V_i) V_i$, and one has

$$|G| = \sum_i \dim(V_i)^2.$$

(the "sum of squares formula").

Proof. By Proposition 2.16, (i) implies (ii), and to prove (i), it is sufficient to show that if V is a finite-dimensional representation of G and $W \subset V$ is any subrepresentation, then there exists a subrepresentation $W' \subset V$ such that $V = W \oplus W'$ as representations.

Choose any complement $\hat{W}$ of W in V. (Thus $V = W \oplus \hat{W}$ as *vector spaces*, but not necessarily as *representations*.) Let P be the projection along $\hat{W}$ onto W, i.e., the operator on V defined by $P|_W = \mathrm{Id}$ and $P|_{\hat{W}} = 0$. Let

$$\overline{P} := \frac{1}{|G|} \sum_{g \in G} \rho(g) P \rho(g^{-1}),$$

where $\rho(g)$ is the action of g on V, and let

$$W' = \ker \overline{P}.$$

Now $\overline{P}|_W = \mathrm{Id}$ and $\overline{P}(V) \subseteq W$, so $\overline{P}^2 = \overline{P}$, so $\overline{P}$ is a projection along W'. Thus, $V = W \oplus W'$ as vector spaces.

Moreover, for any $h \in G$ and any $y \in W'$,

$$\overline{P}\rho(h)y = \frac{1}{|G|} \sum_{g \in G} \rho(g) P \rho(g^{-1}h) y = \frac{1}{|G|} \sum_{\ell \in G} \rho(h\ell) P \rho(\ell^{-1}) y = \rho(h)\overline{P}y = 0,$$

so $\rho(h)y \in \ker \overline{P} = W'$. Thus, W' is invariant under the action of G and is therefore a subrepresentation of V. Thus, $V = W \oplus W'$ is the desired decomposition into subrepresentations. □

The converse to Theorem 3.1(i) also holds.

Proposition 3.2. *If $k[G]$ is semisimple, then the characteristic of k does not divide $|G|$.*

Proof. Write $k[G] = \bigoplus_{i=1}^r \text{End } V_i$, where the V_i are irreducible representations and $V_1 = k$ is the trivial one-dimensional representation. Then

$$k[G] = k \oplus \bigoplus_{i=2}^{r} \text{End } V_i = k \oplus \bigoplus_{i=2}^{r} d_i V_i,$$

where $d_i = \dim V_i$. By Schur's Lemma,

$$\begin{aligned} \text{Hom}_{k[G]}(k, k[G]) &= k\Lambda \\ \text{Hom}_{k[G]}(k[G], k) &= k\epsilon, \end{aligned}$$

for nonzero homomorphisms of representations $\epsilon : k[G] \to k$ and $\Lambda : k \to k[G]$ unique up to scaling. We can take ϵ such that $\epsilon(g) = 1$ for all $g \in G$, and Λ such that $\Lambda(1) = \sum_{g \in G} g$. Then

$$\epsilon \circ \Lambda(1) = \epsilon\left(\sum_{g \in G} g\right) = \sum_{g \in G} 1 = |G|.$$

If $|G| = 0$, then Λ has no left inverse, as $(a\epsilon) \circ \Lambda(1) = 0$ for any $a \in k$. This is a contradiction. □

Example 3.3. If $G = \mathbb{Z}/p\mathbb{Z}$ and k has characteristic p, then every irreducible representation of G over k is trivial (so $k[\mathbb{Z}/p\mathbb{Z}]$ indeed is not semisimple). Indeed, an irreducible representation of this group is a 1-dimensional space, on which the generator acts by a p-th root of unity, and every p-th root of unity in k equals 1, as $x^p - 1 = (x-1)^p$ over k.

Problem 3.4. *Let G be a group of order p^n. Show that every irreducible representation of G over a field k of characteristic p is trivial.*

3.2 Characters

If V is a finite-dimensional representation of a finite group G, then its character $\chi_V : G \to k$ is defined by the formula $\chi_V(g) = \text{tr}|_V(\rho(g))$. Obviously, $\chi_V(g)$ is simply the restriction of the character $\chi_V(a)$ of V as a representation of the algebra $A = k[G]$ to the basis $G \subset A$, so it carries exactly the same information. The character is a *central* or *class function*: $\chi_V(g)$ depends only on the conjugacy class of g; i.e., $\chi_V(hgh^{-1}) = \chi_V(g)$.

Theorem 3.5. *If the characteristic of k does not divide $|G|$, characters of irreducible representations of G form a basis in the space $F_c(G, k)$ of class functions on G.*

Proof. By the Maschke theorem, $k[G]$ is semisimple, so by Theorem 2.17, the characters are linearly independent and are a basis of $(A/[A, A])^*$, where $A = k[G]$. It suffices to note that, as vector spaces over k,

$$\begin{aligned} (A/[A, A])^* &\cong \{\varphi \in \text{Hom}_k(k[G], k) \mid gh - hg \in \ker \varphi \ \forall g, h \in G\} \\ &\cong \{f \in \text{Fun}(G, k) \mid f(gh) = f(hg) \ \forall g, h \in G\}, \end{aligned}$$

which is precisely $F_c(G, k)$. □

Corollary 3.6. *The number of isomorphism classes of irreducible representations of G equals the number of conjugacy classes of G (if $|G| \neq 0$ in k).*

Exercise. Show that if $|G| = 0$ in k then the number of isomorphism classes of irreducible representations of G over k is strictly less than the number of conjugacy classes in G.

Hint. Let $P = \sum_{g\in G} g \in k[G]$. Then $P^2 = 0$. So P has zero trace in every finite dimensional representation of G over k.

Corollary 3.7. *Any representation of G is determined by its character if k has characteristic 0; namely, $\chi_V = \chi_W$ implies $V \cong W$.*

3.3 Examples

The following are examples of representations of finite groups over $\mathbb{C}$.

1. Finite abelian groups $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$. Let $G^\vee$ be the set of irreducible representations of G. Every element of G forms a conjugacy class, so $|G^\vee| = |G|$. Recall that all irreducible representations over $\mathbb{C}$ (and algebraically closed fields in general) of commutative algebras and groups are one-dimensional. Thus, $G^\vee$ is an abelian group: if $\rho_1, \rho_2 : G \to \mathbb{C}^\times$ are irreducible representations then so are $\rho_1(g)\rho_2(g)$ and $\rho_1(g)^{-1}$. $G^\vee$ is called the *dual* or *character group* of G.

   For given $n \geq 1$, define $\rho : \mathbb{Z}_n \to \mathbb{C}^\times$ by $\rho(m) = e^{2\pi i m/n}$. Then $\mathbb{Z}_n^\vee = \{\rho^k : k = 0, \ldots, n-1\}$, so $\mathbb{Z}_n^\vee \cong \mathbb{Z}_n$. In general,

   $$(G_1 \times G_2 \times \cdots \times G_n)^\vee = G_1^\vee \times G_2^\vee \times \cdots \times G_n^\vee,$$

   so $G^\vee \cong G$ for any finite abelian group G. This isomorphism is, however, noncanonical: the particular decomposition of G as $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ is not unique as far as which elements of G correspond to $\mathbb{Z}_{n_1}$, etc. is concerned. On the other hand, $G \cong (G^\vee)^\vee$ is a canonical isomorphism, given by $\varphi : G \to (G^\vee)^\vee$, where $\varphi(g)(\chi) = \chi(g)$.

2. The symmetric group S_3. In S_n, conjugacy classes are determined by cycle decomposition sizes: two permutations are conjugate if and only if they have the same number of cycles of each length. For S_3, there are 3 conjugacy classes, so there are 3 different irreducible representations over $\mathbb{C}$. If their dimensions are d_1, d_2, d_3, then $d_1^2+d_2^2+d_3^2 = 6$, so S_3 must have two 1-dimensional and one 2-dimensional representations. The 1-dimensional representations are the trivial representation $\mathbb{C}_+$ given by $\rho(\sigma) = 1$ and the sign representation $\mathbb{C}_-$ given by $\rho(\sigma) = (-1)^\sigma$.

   The 2-dimensional representation can be visualized as representing the symmetries of the equilateral triangle with vertices 1, 2, 3 at the points $(\cos 120°, \sin 120°)$, $(\cos 240°, \sin 240°)$, $(1, 0)$ of the coordinate plane, respectively. Thus, for example,

   $$\rho((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho((123)) = \begin{pmatrix} \cos 120° & -\sin 120° \\ \sin 120° & \cos 120° \end{pmatrix}.$$

   To show that this representation is irreducible, consider any subrepresentation V. V must be the span of a subset of the eigenvectors of $\rho((12))$, which are the nonzero multiples of $(1, 0)$ and $(0, 1)$. V must also be the span of a subset of the eigenvectors of $\rho((123))$, which are different vectors. Thus, V must be either $\mathbb{C}^2$ or 0.

3. The quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, with defining relations

   $$i = jk = -kj, \qquad j = ki = -ik, \qquad k = ij = -ji, \qquad -1 = i^2 = j^2 = k^2.$$

The 5 conjugacy classes are $\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}$, so there are 5 different irreducible representations, the sum of the squares of whose dimensions is 8, so their dimensions must be 1, 1, 1, 1, and 2.

The center $Z(Q_8)$ is $\{\pm 1\}$, and $Q_8/Z(Q_8) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The four 1-dimensional irreducible representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$ can be "pulled back" to Q_8. That is, if $q : Q_8 \to Q_8/Z(Q_8)$ is the quotient map, and ρ any representation of $Q_8/Z(Q_8)$, then $\rho \circ q$ gives a representation of Q_8.

The 2-dimensional representation is $V = \mathbb{C}^2$, given by $\rho(-1) = -\mathrm{Id}$ and

$$\rho(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho(j) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \qquad \rho(k) = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}. \tag{3}$$

These are the Pauli matrices, which arise in quantum mechanics.

Exercise. Show that the 2-dimensional irreducible representation of Q_8 can be realized in the space of functions $f : Q_8 \to \mathbb{C}$ such that $f(gi) = \sqrt{-1}f(g)$ (the action of G is by right multiplication, $g \circ f(x) = f(xg)$).

4. The symmetric group S_4. The order of S_4 is 24, and there are 5 conjugacy classes: $e, (12), (123), (1234), (12)(34)$. Thus the sum of the squares of the dimensions of 5 irreducible representations is 24. As with S_3, there are two of dimension 1: the trivial and sign representations, $\mathbb{C}_+$ and $\mathbb{C}_-$. The other three must then have dimensions 2, 3, and 3. Because $S_3 \cong S_4/\mathbb{Z}_2 \times \mathbb{Z}_2$, where $\mathbb{Z}_2 \times \mathbb{Z}_2$ is $\{e, (12)(34), (13)(24), (14)(23)\}$, the 2-dimensional representation of S_3 can be pulled back to the 2-dimensional representation of S_4, which we will call $\mathbb{C}^2$.

   We can consider S_4 as the group of rotations of a cube acting by permuting the interior diagonals (or, equivalently, on a regular octahedron permuting pairs of opposite faces); this gives the 3-dimensional representation $\mathbb{C}^3_+$.

   The last 3-dimensional representation is $\mathbb{C}^3_-$, the product of $\mathbb{C}^3_+$ with the sign representation. $\mathbb{C}^3_+$ and $\mathbb{C}^3_-$ are different, for if g is a transposition, $\det g|_{\mathbb{C}^3_+} = 1$ while $\det g|_{\mathbb{C}^3_-} = (-1)^3 = -1$. Note that another realization of $\mathbb{C}^3_-$ is by action of S_4 by symmetries (not necessarily rotations) of the regular tetrahedron. Yet another realization of this representation is the space of functions on the set of 4 elements (on which S_4 acts by permutations) with zero sum of values.

3.4 Duals and tensor products of representations

If V is a representation of a group G, then V^* is also a representation, via

$$\rho_{V^*}(g) = (\rho_V(g)^*)^{-1} = (\rho_V(g)^{-1})^* = \rho_V(g^{-1})^*.$$

The character is $\chi_{V^*}(g) = \chi_V(g^{-1})$.

We have $\chi_V(g) = \sum \lambda_i$, where the λ_i are the eigenvalues of g in V. These eigenvalues must be roots of unity because $\rho(g)^{|G|} = \rho(g^{|G|}) = \rho(e) = \mathrm{Id}$. Thus for complex representations

$$\chi_{V^*}(g) = \chi_V(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\sum \lambda_i} = \overline{\chi_V(g)}.$$

In particular, $V \cong V^*$ as *representations* (not just as vector spaces) if and only if $\chi_V(g) \in \mathbb{R}$ for all $g \in G$.

If V, W are representations of G, then $V \otimes W$ is also a representation, via

$$\rho_{V \otimes W}(g) = \rho_V(g) \otimes \rho_W(g).$$

Therefore, $\chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g)$.

An interesting problem discussed below is to decompose $V \otimes W$ (for irreducible V, W) into the direct sum of irreducible representations.

3.5 Orthogonality of characters

We define a positive definite Hermitian inner product on $\mathrm{F}_c(G, \mathbb{C})$ (the space of central functions) by

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}.$$

The following theorem says that characters of irreducible representations of G form an orthonormal basis of $F_c(G, \mathbb{C})$ under this inner product.

Theorem 3.8. *For any representations V, W*

$$(\chi_V, \chi_W) = \dim Hom_G(W, V),$$

and

$$(\chi_V, \chi_W) = \begin{cases} 1, \text{ if } V \cong W, \\ 0, \text{ if } V \ncong W \end{cases}$$

if V, W are irreducible.

Proof. By the definition

$$\begin{aligned} (\chi_V, \chi_W) &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\overline{\chi_W(g)} = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\chi_{W^*}(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{V \otimes W^*}(g) = \mathrm{Tr}\,|_{V \otimes W^*}(P), \end{aligned}$$

where $P = \frac{1}{|G|} \sum_{g \in G} g \in Z(\mathbb{C}[G])$. (Here $Z(\mathbb{C}[G])$ denotes the center of $\mathbb{C}[G]$). If X is an irreducible representation of G then

$$P|_X = \begin{cases} \mathrm{Id}, \text{ if } X = \mathbb{C}, \\ 0, \quad X \neq \mathbb{C}. \end{cases}$$

Therefore, for any representation X the operator $P|_X$ is the G-invariant projector onto the subspace X^G of G-invariants in X. Thus,

$$\begin{aligned} \mathrm{Tr}\,|_{V \otimes W^*}(P) &= \dim \mathrm{Hom}_G(\mathbb{C}, V \otimes W^*) \\ &= \dim(V \otimes W^*)^G = \dim \mathrm{Hom}_G(W, V). \end{aligned}$$

□

Theorem 3.8 gives a powerful method of checking if a given complex representation V of a finite group G is irreducible. Indeed, it implies that V is irreducible if and only if $(\chi_V, \chi_V) = 1$.

Exercise. Let G be a finite group. Let V_i be the irreducible complex representations of G.

For every i, let

$$\psi_i = \frac{\dim V_i}{|G|} \sum_{g \in G} \chi_{V_i}(g) \cdot g^{-1} \in \mathbb{C}[G].$$

(i) Prove that ψ_i acts on V_j as the identity if $j = i$, and as the null map if $j \neq i$.

(ii) Prove that ψ_i are **idempotents**, i.e., $\psi_i^2 = \psi_i$ for any i, and $\psi_i\psi_j = 0$ for any $i \neq j$.

Hint: In **(i)**, notice that ψ_i commutes with any element of $k[G]$, and thus acts on V_j as an intertwining operator. Corollary 1.17 thus yields that ψ_i acts on V_j as a scalar. Compute this scalar by taking its trace in V_j.

Here is another "orthogonality formula" for characters, in which summation is taken over irreducible representations rather than group elements.

Theorem 3.9. *Let $g, h \in G$, and let Z_g denote the centralizer of g in G. Then*

$$\sum_V \chi_V(g)\overline{\chi_V(h)} = \begin{cases} |Z_g| & \text{if } g \text{ is conjugate to } h \\ 0, & \text{otherwise} \end{cases}$$

where the summation is taken over all irreducible representations of G.

Proof. As noted above, $\overline{\chi_V(h)} = \chi_{V^*}(h)$, so the left hand side equals (using Maschke's theorem):

$$\sum_V \chi_V(g)\chi_{V^*}(h) = \mathrm{Tr}|_{\oplus_V V \otimes V^*}(g \otimes (h^*)^{-1}) =$$

$$\mathrm{Tr}|_{\oplus_V \mathrm{End} V}(x \mapsto gxh^{-1}) = \mathrm{Tr}|_{\mathbb{C}[G]}(x \mapsto gxh^{-1}).$$

If g and h are not conjugate, this trace is clearly zero, since the matrix of the operator $x \mapsto gxh^{-1}$ in the basis of group elements has zero diagonal entries. On the other hand, if g and h are in the same conjugacy class, the trace is equal to the number of elements x such that $x = gxh^{-1}$, i.e., the order of the centralizer Z_g of g. We are done. $\square$

Remark. Another proof of this result is as follows. Consider the matrix U whose rows are labeled by irreducible representations of G and columns by conjugacy classes, with entries $U_{V,g} = \chi_V(g)/\sqrt{|Z_g|}$. Note that the conjugacy class of g is G/Z_g, thus $|G|/|Z_g|$ is the number of elements conjugate to G. Thus, by Theorem 3.8, the rows of the matrix U are orthonormal. This means that U is unitary and hence its columns are also orthonormal, which implies the statement.

3.6 Unitary representations. Another proof of Maschke's theorem for complex representations

Definition 3.10. A unitary finite dimensional representation of a group G is a representation of G on a complex finite dimensional vector space V over $\mathbb{C}$ equipped with a G-invariant positive definite Hermitian form[4] $(,)$, i.e., such that $\rho_V(g)$ are unitary operators: $(\rho_V(g)v, \rho_V(g)w) = (v, w)$.

[4]We agree that Hermitian forms are linear in the first argument and antilinear in the second one.

Theorem 3.11. *If G is finite, then any finite dimensional representation of G has a unitary structure. If the representation is irreducible, this structure is unique up to scaling by a positive real number.*

Proof. Take any positive definite form B on V and define another form $\overline{B}$ as follows:

$$\overline{B}(v,w) = \sum_{g\in G} B(\rho_V(g)v, \rho_V(g)w)$$

Then $\overline{B}$ is a positive definite Hermitian form on V, and $\rho_V(g)$ are unitary operators. If V is an irreducible representation and B_1, B_2 are two positive definite Hermitian forms on V, then $B_1(v,w) = B_2(Av,w)$ for some homomorphism $A : V \to V$ (since any positive definite Hermitian form is nondegenerate). By Schur's lemma, $A = \lambda\mathrm{Id}$, and clearly $\lambda > 0$. □

Theorem 3.11 implies that if V is a finite dimensional representation of a finite group G, then the *complex conjugate representation* $\overline{V}$ (i.e., the same space V with the same addition and the same action of G, but complex conjugate action of scalars) is isomorphic to the dual representation V^*. Indeed, a homomorphism of representations $\overline{V} \to V^*$ is obviously the same thing as an invariant sesquilinear form on V (i.e. a form additive on both arguments which is linear on the first one and antilinear on the second one), and an isomorphism is the same thing as a nondegenerate invariant sesquilinear form. So one can use a unitary structure on V to define an isomorphism $\overline{V} \to V^*$.

Theorem 3.12. *A finite dimensional unitary representation V of any group G is completely reducible.*

Proof. Let W be a subrepresentation of V. Let $W^\perp$ be the orthogonal complement of W in V under the Hermitian inner product. Then $W^\perp$ is a subrepresentation of W, and $V = W \oplus W^\perp$. This implies that V is completely reducible. □

Theorems 3.11 and 3.12 imply Maschke's theorem for complex representations (Theorem 3.1). Thus, we have obtained a new proof of this theorem over the field of complex numbers.

Remark 3.13. Theorem 3.12 shows that for infinite groups G, a finite dimensional representation may fail to admit a unitary structure (as there exist finite dimensional representations, e.g. for $G = \mathbb{Z}$, which are indecomposable but not irreducible).

3.7 Orthogonality of matrix elements

Let V be an irreducible representation of a finite group G, and $v_1, v_2, \ldots, v_n$ be an orthonormal basis of V under the invariant Hermitian form. The matrix elements of V are $t_{ij}^V(x) = (\rho_V(x)v_i, v_j)$.

Proposition 3.14. *(i) Matrix elements of nonisomorphic irreducible representations are orthogonal in* $\mathrm{Fun}(G,\mathbb{C})$ *under the form* $(f,g) = \frac{1}{|G|}\sum_{x\in G} f(x)\overline{g(x)}$.

(ii) $(t_{ij}^V, t_{i'j'}^V) = \delta_{ii'}\delta_{jj'} \cdot \frac{1}{\dim V}$

Thus, matrix elements of irreducible representations of G form an orthogonal basis of $\mathrm{Fun}(G,\mathbb{C})$.

Proof. Let V and W be two irreducible representations of G. Take $\{v_i\}$ to be an orthonormal basis of V and $\{w_i\}$ to be an orthonormal basis of W under their positive definite invariant Hermitian forms. Let $w_i^* \in W^*$ be the linear function on W defined by taking the inner product with

w_i: $w_i^*(u) = (u, w_i)$. Then for $x \in G$ we have $(xw_i^*, w_j^*) = \overline{(xw_i, w_j)}$. Therefore, putting $P = \frac{1}{|G|}\sum_{x\in G} x$, we have

$$(t_{ij}^V, t_{i'j'}^W) = |G|^{-1}\sum_{x\in G}(xv_i, v_j)\overline{(xw_{i'}, w_{j'})} = |G|^{-1}\sum_{x\in G}(xv_i, v_j)(xw_{i'}^*, w_{j'}^*) = (P(v_i \otimes w_{i'}^*), v_j \otimes w_{j'}^*)$$

If $V = W$, this is zero, since P projects to the trivial representation, which does not occur in $V \otimes W^*$. If $V = W$, we need to consider $(P(v_i \otimes v_{i'}^*), v_j \otimes v_{j'}^*)$. We have a G-invariant decomposition

$$\begin{aligned} V \otimes V^* &= \mathbb{C} \oplus L \\ \mathbb{C} &= \text{span}(\sum v_k \otimes v_k^*) \\ L &= \text{span}_{a:\sum_k a_{kk}=0}(\sum_{k,l} a_{kl} v_k \otimes v_l^*), \end{aligned}$$

and P projects to the first summand along the second one. The projection of $v_i \otimes v_{i'}^*$ to $\mathbb{C} \subset \mathbb{C} \oplus L$ is thus

$$\frac{\delta_{ii'}}{\dim V}\sum v_k \otimes v_k^*$$

This shows that

$$(P(v_i \otimes v_{i'}^*), v_j \otimes v_{j'}^*) = \frac{\delta_{ii'}\delta_{jj'}}{\dim V}$$

which finishes the proof of (i) and (ii). The last statement follows immediately from the sum of squares formula. □

3.8 Character tables, examples

The characters of all the irreducible representations of a finite group can be arranged into a character table, with conjugacy classes of elements as the columns, and characters as the rows. More specifically, the first row in a character table lists representatives of conjugacy classes, the second one the numbers of elements in the conjugacy classes, and the other rows list the values of the characters on the conjugacy classes. Due to Theorems 3.8 and 3.9 the rows and columns of a character table are orthonormal with respect to the appropriate inner products.

Note that in any character table, the row corresponding to the trivial representation consists of ones, and the column corresponding to the neutral element consists of the dimensions of the representations.

Here is, for example, the character table of S_3 :

S_3	Id	(12)	(123)
#	1	3	2
$\mathbb{C}_+$	1	1	1
$\mathbb{C}_-$	1	-1	1
$\mathbb{C}^2$	2	0	-1

It is obtained by explicitly computing traces in the irreducible representations.

For another example consider A_4, the group of even permutations of 4 items. There are three one-dimensional representations (as A_4 has a normal subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and $A_4/\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \mathbb{Z}_3$). Since there are four conjugacy classes in total, there is one more irreducible representation of dimension 3. Finally, the character table is

A_4	Id	(123)	(132)	(12)(34)
#	1	4	4	3
$\mathbb{C}$	1	1	1	1
$\mathbb{C}_\epsilon$	1	ϵ	ϵ^2	1
$\mathbb{C}_{\epsilon^2}$	1	ϵ^2	ϵ	1
$\mathbb{C}^3$	3	0	0	-1

where $\epsilon = \exp(\frac{2\pi i}{3})$.

The last row can be computed using the orthogonality of rows. Another way to compute the last row is to note that $\mathbb{C}^3$ is the representation of A_4 by rotations of the regular tetrahedron: in this case $(123), (132)$ are the rotations by 120^0 and 240^0 around a perpendicular to a face of the tetrahedron, while $(12)(34)$ is the rotation by 180^0 around an axis perpendicular to two opposite edges.

Example 3.15. The following three character tables are of Q_8, S_4, and A_5 respectively.

Q_8	1	-1	i	j	k
#	1	1	2	2	2
$\mathbb{C}_{++}$	1	1	1	1	1
$\mathbb{C}_{+-}$	1	1	1	-1	-1
$\mathbb{C}_{-+}$	1	1	-1	1	-1
$\mathbb{C}_{--}$	1	1	-1	-1	1
$\mathbb{C}^2$	2	-2	0	0	0

S_4	Id	(12)	(12)(34)	(123)	(1234)
#	1	6	3	8	6
$\mathbb{C}_+$	1	1	1	1	1
$\mathbb{C}_-$	1	-1	1	1	-1
$\mathbb{C}^2$	2	0	2	-1	0
$\mathbb{C}^3_+$	3	-1	-1	0	1
$\mathbb{C}^3_-$	3	1	-1	0	-1

A_5	Id	(123)	(12)(34)	(12345)	(13245)
#	1	20	15	12	12
$\mathbb{C}$	1	1	1	1	1
$\mathbb{C}^3_+$	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
$\mathbb{C}^3_-$	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
$\mathbb{C}^4$	4	1	0	-1	-1
$\mathbb{C}^5$	5	-1	1	0	0

Indeed, the computation of the characters of the 1-dimensional representations is straightforward.

The character of the 2-dimensional representation of Q_8 is obtained from the explicit formula (3) for this representation, or by using the orthogonality.

For S_4, the 2-dimensional irreducible representation is obtained from the 2-dimensional irreducible representation of S_3 via the surjective homomorphism $S_4 \to S_3$, which allows to obtain its character from the character table of S_3.

The character of the 3-dimensional representation $\mathbb{C}^3_+$ is computed from its geometric realization by rotations of the cube. Namely, by rotating the cube, S_4 permutes the main diagonals. Thus (12) is the rotation by 180^0 around an axis that is perpendicular to two opposite edges, $(12)(34)$

is the rotation by 180^0 around an axis that is perpendicular to two opposite faces, (123) is the rotation around a main diagonal by 120^0, and (1234) is the rotation by 90^0 around an axis that is perpendicular to two opposite faces; this allows us to compute the traces easily, using the fact that the trace of a rotation by the angle ϕ in $\mathbb{R}^3$ is $1 + 2\cos\phi$. Now the character of $\mathbb{C}^3_-$ is found by multiplying the character of $\mathbb{C}^3_+$ by the character of the sign representation.

Finally, we explain how to obtain the character table of A_5 (even permutations of 5 items). The group A_5 is the group of rotations of the regular icosahedron. Thus it has a 3-dimensional "rotation representation" $\mathbb{C}^3_+$, in which (12)(34) is the rotation by 180^0 around an axis perpendicular to two opposite edges, (123) is the rotation by 120^0 around an axis perpendicular to two opposite faces, and (12345), (13254) are the rotations by 72^0, respectively 144^0, around axes going through two opposite vertices. The character of this representation is computed from this description in a straightforward way.

Another representation of A_5, which is also 3-dimensional, is $\mathbb{C}^3_+$ twisted by the automorphism of A_5 given by conjugation by (12) inside S_5. This representation is denoted by $\mathbb{C}^3_-$. It has the same character as $\mathbb{C}^3_+$, except that the conjugacy classes (12345) and (13245) are interchanged.

There are two remaining irreducible representations, and by the sum of squares formula their dimensions are 4 and 5. So we call them $\mathbb{C}^4$ and $\mathbb{C}^5$.

The representation $\mathbb{C}^4$ is realized on the space of functions on the set $\{1,2,3,4,5\}$ with zero sum of values, where A_5 acts by permutations (check that it is irreducible!). The character of this representation is equal to the character of the 5-dimensional permutation representation minus the character of the 1-dimensional trivial representation (constant functions). The former at an element g equals to the number of items among 1,2,3,4,5 which are fixed by g.

The representation $\mathbb{C}^5$ is realized on the space of functions on pairs of opposite vertices of the icosahedron which has zero sum of values (check that it is irreducible!). The character of this representation is computed similarly to the character of $\mathbb{C}^4$, or from the orthogonality formula.

3.9 Computing tensor product multiplicities using character tables

Character tables allow us to compute the tensor product multiplicities N_{ij}^k using

$$V_i \otimes V_j = \sum N_{ij}^k V_k, \quad N_{ij}^k = (\chi_i \chi_j, \chi_k)$$

Example 3.16. The following tables represent computed tensor product multiplicities of irreducible representations of S_3, S_4, and A_5 respectively.

S_3	$\mathbb{C}_+$	$\mathbb{C}_-$	$\mathbb{C}^2$
$\mathbb{C}_+$	$\mathbb{C}_+$	$\mathbb{C}_-$	$\mathbb{C}^2$
$\mathbb{C}_-$		$\mathbb{C}_+$	$\mathbb{C}^2$
$\mathbb{C}^2$			$\mathbb{C}_+ \oplus \mathbb{C}_- \oplus \mathbb{C}^2$

S_4	$\mathbb{C}_+$	$\mathbb{C}_-$	$\mathbb{C}^2$	$\mathbb{C}^3_+$	$\mathbb{C}^3_-$
$\mathbb{C}_+$	$\mathbb{C}_+$	$\mathbb{C}_-$	$\mathbb{C}^2$	$\mathbb{C}^3_+$	$\mathbb{C}^3_-$
$\mathbb{C}_-$		$\mathbb{C}_+$	$\mathbb{C}^2$	$\mathbb{C}^3_-$	$\mathbb{C}^3_+$
$\mathbb{C}^2$			$\mathbb{C}_+ \oplus \mathbb{C}_- \oplus \mathbb{C}^2$	$\mathbb{C}^3_+ \oplus \mathbb{C}^3_-$	$\mathbb{C}^3_+ \oplus \mathbb{C}^3_-$
$\mathbb{C}^3_+$				$\mathbb{C}_+ \oplus \mathbb{C}^2 \oplus \mathbb{C}^3_+ \oplus \mathbb{C}^3_-$	$\mathbb{C}_- \oplus \mathbb{C}^2 \oplus \mathbb{C}^3_+ \oplus \mathbb{C}^3_-$
$\mathbb{C}^3_-$					$\mathbb{C}_+ \oplus \mathbb{C}^2 \oplus \mathbb{C}^3_+ \oplus \mathbb{C}^3_-$

A_5	$\mathbb{C}$	$\mathbb{C}^3_+$	$\mathbb{C}^3_-$	$\mathbb{C}^4$	$\mathbb{C}^5$
$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}^3_+$	$\mathbb{C}^3_-$	$\mathbb{C}^4$	$\mathbb{C}^5$
$\mathbb{C}^3_+$		$\mathbb{C}\oplus\mathbb{C}^5\oplus\mathbb{C}^3_+$	$\mathbb{C}^4\oplus\mathbb{C}^5$	$\mathbb{C}^3_-\oplus\mathbb{C}^4\oplus\mathbb{C}^5$	$\mathbb{C}^3_+\oplus\mathbb{C}^3_-\oplus\mathbb{C}^4\oplus\mathbb{C}^5$
$\mathbb{C}^3_-$			$\mathbb{C}\oplus\mathbb{C}^5\oplus\mathbb{C}^3_+$	$\mathbb{C}^3_+\oplus\mathbb{C}^4\oplus\mathbb{C}^5$	$\mathbb{C}^3_+\oplus\mathbb{C}^3_-\oplus\mathbb{C}^4\oplus\mathbb{C}^5$
$\mathbb{C}^4$				$\mathbb{C}^3_+\oplus\mathbb{C}^3_-\oplus\mathbb{C}\oplus\mathbb{C}^4\oplus\mathbb{C}^5$	$\mathbb{C}^3_+\oplus\mathbb{C}^3_-\oplus 2\mathbb{C}^5\oplus\mathbb{C}^4$
$\mathbb{C}^5$					$\mathbb{C}\oplus\mathbb{C}^3_+\oplus\mathbb{C}^3_-\oplus 2\mathbb{C}^4\oplus 2\mathbb{C}^5$

3.10 Problems

Problem 3.17. *Let G be the group of symmetries of a regular N-gon (it has $2N$ elements).*

(a) Describe all irreducible complex representations of this group (consider the cases of odd and even N)

(b) Let V be the 2-dimensional complex representation of G obtained by complexification of the standard representation on the real plane (the plane of the polygon). Find the decomposition of $V \otimes V$ in a direct sum of irreducible representations.

Problem 3.18. *Let G be the group of 3 by 3 matrices over $\mathbb{F}_p$ which are upper triangular and have ones on the diagonal, under multiplication (its order is p^3). It is called the Heisenberg group. For any complex number z such that $z^p = 1$ we define a representation of G on the space V of complex functions on $\mathbb{F}_p$, by*

$$(\rho \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} f)(x) = f(x-1),$$

$$(\rho \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} f)(x) = z^x f(x).$$

(note that z^x makes sense since $z^p = 1$).

(a) Show that such a representation exists and is unique, and compute $\rho(g)$ for all $g \in G$.

(b) Denote this representation by R_z. Show that R_z is irreducible if and only if $z = 1$.

(c) Classify all 1-dimensional representations of G. Show that R_1 decomposes into a direct sum of 1-dimensional representations, where each of them occurs exactly once.

(d) Use (a)-(c) and the "sum of squares" formula to classify all irreducible representations of G.

Problem 3.19. *Let V be a finite dimensional complex vector space, and $GL(V)$ be the group of invertible linear transformations of V. Then S^nV and Λ^mV ($m \le dim(V)$) are representations of $GL(V)$ in a natural way. Show that they are irreducible representations.*

Hint: Choose a basis $\{e_i\}$ in V. Find a diagonal element H of $GL(V)$ such that $\rho(H)$ has distinct eigenvalues. (where ρ is one of the above representations). This shows that if W is a subrepresentation, then it is spanned by a subset S of a basis of eigenvectors of $\rho(H)$. Use the invariance of W under the operators $\rho(1+E_{ij})$ (where E_{ij} is defined by $E_{ij}e_k = \delta_{jk}e_i$) for all $i = j$ to show that if the subset S is nonempty, it is necessarily the entire basis.

Problem 3.20. *Recall that the adjacency matrix of a graph Γ (without multiple edges) is the matrix in which the ij-th entry is 1 if the vertices i and j are connected with an edge, and zero otherwise. Let Γ be a finite graph whose automorphism group is nonabelian. Show that the adjacency matrix of Γ must have repeated eigenvalues.*

Problem 3.21. *Let I be the set of vertices of a regular icosahedron ($|I| = 12$). Let* $\mathrm{Fun}(I)$ *be the space of complex functions on I. Recall that the group $G = A_5$ of even permutations of 5 items acts on the icosahedron, so we have a 12-dimensional representation of G on* $\mathrm{Fun}(I)$.

(a) Decompose this representation in a direct sum of irreducible representations (i.e., find the multiplicities of occurrence of all irreducible representations).

(b) Do the same for the representation of G on the space of functions on the set of faces and the set of edges of the icosahedron.

Problem 3.22. *Let $\mathbb{F}_q$ be a finite field with q elements, and G be the group of nonconstant inhomogeneous linear transformations, $x \to ax + b$, over $\mathbb{F}_q$ (i.e., $a \in \mathbb{F}_q^\times, b \in \mathbb{F}_q$). Find all irreducible complex representations of G, and compute their characters. Compute the tensor products of irreducible representations.*

Hint. Let V be the representation of G on the space of functions on $\mathbb{F}_q$ with sum of all values equal to zero. Show that V is an irreducible representation of G.

Problem 3.23. *Let $G = SU(2)$ (unitary 2 by 2 matrices with determinant 1), and $V = \mathbb{C}$ the standard 2-dimensional representation of $SU(2)$. We consider V as a real representation, so it is 4-dimensional.*

(a) Show that V is irreducible (as a real representation).

(b) Let $\mathbb{H}$ be the subspace of $\mathrm{End}_{\mathbb{R}}(V)$ *consisting of endomorphisms of V as a real representation. Show that $\mathbb{H}$ is 4-dimensional and closed under multiplication. Show that every nonzero element in $\mathbb{H}$ is invertible, i.e., $\mathbb{H}$ is an algebra with division.*

(c) Find a basis $1, i, j, k$ of $\mathbb{H}$ such that 1 is the unit and $i^2 = j^2 = k^2 = -1$, $ij = -ji = k, jk = -kj = i, ki = -ik = j$. Thus we have that Q_8 is a subgroup of the group $\mathbb{H}^\times$ of invertible elements of $\mathbb{H}$ under multiplication.

The algebra $\mathbb{H}$ is called the quaternion algebra.

(d) For $q = a+bi+cj+dk$, $a, b, c, d \in \mathbb{R}$, let $\bar{q} = a-bi-cj-dk$, and $||q||^2 = q\bar{q} = a^2+b^2+c^2$. Show that $\overline{q_1 q_2} = \bar{q}_2 \bar{q}_1$, and $||q_1 q_2|| = ||q_1|| \cdot ||q_2||$.

(e) Let G be the group of quaternions of norm 1. Show that this group is isomorphic to $SU(2)$. (Thus geometrically $SU(2)$ is the 3-dimensional sphere).

(f) Consider the action of G on the space $V \subset \mathbb{H}$ spanned by i, j, k, by $x \to qxq^{-1}$, q G, $x \in V$. Since this action preserves the norm on V, we have a homomorphism $h : SU(2) \to SO(3)$, where $SO(3)$ is the group of rotations of the three-dimensional Euclidean space. Show that this homomorphism is surjective and that its kernel is $\{1, -1\}$.

Problem 3.24. *It is known that the classification of finite subgroups of $SO(3)$ is as follows:*

1) the cyclic group $\mathbb{Z}/n\mathbb{Z}$, $n \geq 1$, generated by a rotation by $2\pi/n$ around an axis;

2) the dihedral group D_n of order $2n$, $n \geq 2$ (the group of rotational symmetries in 3-space of a plane containing a regular n-gon[5]*;*

3) the group of rotations of the regular tetrahedron (A_4).

4) the group of rotations of the cube or regular octahedron (S_4).

5) the group of rotations of a regular dodecahedron or icosahedron (A_5).

[5] A regular 2-gon is just a line segment.

(a) Derive this classification.

Hint. Let G be a finite subgroup of $SO(3)$. Consider the action of G on the unit sphere. A point of the sphere preserved by some nontrivial element of G is called a pole. Show that every nontrivial element of G fixes a unique pair of opposite poles, and that the subgroup of G fixing a particular pole P is cyclic, of some order m (called the order of P). Thus the orbit of P has n/m elements, where $n = |G|$. Now let $P_1, ..., P_k$ be the poles representing all the orbits of G on the set of poles, and $m_1, ..., m_k$ be their orders. By counting nontrivial elements of G, show that

$$2(1 - \frac{1}{n}) = \sum_i (1 - \frac{1}{m_i}).$$

Then find all possible m_i and n that can satisfy this equation and classify the corresponding groups.

(b) Using this classification, classify finite subgroups of $SU(2)$ (use the homomorphism $SU(2) \to SO(3)$).

Problem 3.25. *Find the characters and tensor products of irreducible complex representations of the Heisenberg group from Problem 3.18.*

Problem 3.26. *Let G be a finite group, and V a complex representation of G which is faithful, i.e., the corresponding map $G \to GL(V)$ is injective. Show that any irreducible representation of G occurs inside S^nV (and hence inside $V^{\otimes n}$) for some n.*

Hint. Show that there exists a vector $u \in V^*$ whose stabilizer in G is 1. Now define the map $SV \to \mathrm{Fun}(G, \mathbb{C})$ sending a polynomial f on V^* to the function f_u on G given by $f_u(g) = f(gu)$. Show that this map is surjective and use this to deduce the desired result.

Problem 3.27. *This problem is about an application of representation theory to physics (elasticity theory). We first describe the physical motivation and then state the mathematical problem.*

Imagine a material which occupies a certain region U in the physical space $V = \mathbb{R}^3$ (a space with a positive definite inner product). Suppose the material is deformed. This means, we have applied a diffeomorphism (=change of coordinates) $g : U \to U'$. The question in elasticity theory is how much stress in the material this deformation will cause.

For every point $P \in U$, let $A_P : V \to V$ be defined by $A_P = dg(P)$. A_P is nondegenerate, so it has a polar decomposition $A_P = D_P O_P$, where O_P is orthogonal and D_P is symmetric. The matrix O_P characterizes the rotation part of A_P (which clearly produces no stress), and D_P is the distortion part, which actually causes stress. If the deformation is small, D_P is close to 1, so $D_P = 1 + d_P$, where d_P is a small symmetric matrix, i.e., an element of S^2V. This matrix is called the deformation tensor at P.

Now we define the stress tensor, which characterizes stress. Let v be a small nonzero vector in V, and σ a small disk perpendicular to v centered at P of area $||v||$. Let F_v be the force with which the part of the material on the v-side of σ acts on the part on the opposite side. It is easy to deduce from Newton's laws that F_v is linear in v, so there exists a linear operator $S_P : V \to V$ such that $F_v = S_P v$. It is called the stress tensor.

An elasticity law is an equation $S_P = f(d_P)$, where f is a function. The simplest such law is a linear law (Hooke's law): $f : S^2V \to End(V)$ is a linear function. In general, such a function is defined by $9 \cdot 6 = 54$ parameters, but we will show there are actually only two essential ones – the compression modulus K and the shearing modulus μ. For this purpose we will use representation theory.

Recall that the group $SO(3)$ of rotations acts on V, so S^2V, $End(V)$ are representations of this group. The laws of physics must be invariant under this group (Galileo transformations), so f must be a homomorphism of representations.

(a) Show that $End(V)$ admits a decomposition $\mathbb{R} \oplus V \oplus W$, where $\mathbb{R}$ is the trivial representation, V is the standard 3-dimensional representation, and W is a 5-dimensional representation of SO(3). Show that $S^2V = \mathbb{R} \oplus W$

(b) Show that V and W are irreducible, even after complexification. Deduce using Schur's lemma that S_P is always symmetric, and for $x \in \mathbb{R}, y \in W$ one has $f(x + y) = Kx + \mu y$ for some real numbers K, μ.

In fact, it is clear from physics that K, μ are positive. Physically, the compression modulus K characterises resistance of the material to compression or dilation, while the shearing modulus μ characterizes its resistance to changing the shape of the object without changing its volume. For instance, clay (used for sculpting) has a large compression modulus but a small shearing modulus.