7 Structure of finite dimensional algebras

In this section we return to studying the structure of finite dimensional algebras. Throughout the section, we work over an algebraically closed field k (of any characteristic).

7.1 Projective modules

Let A be an algebra, and P be a left A-module.

Theorem 7.1. *The following properties of P are equivalent:*

(i) If $\alpha : M \to N$ is a surjective morphism, and $\nu : P \to N$ any morphism, then there exists a morphism $\mu : P \to M$ such that $\alpha \circ \mu = \nu$.

(ii) Any surjective morphism $\alpha : M \to P$ splits, i.e., there exists $\mu : P \to M$ such that $\alpha \circ \mu = \mathrm{id}$.

(iii) There exists another A-module Q such that $P \oplus Q$ is a free A-module, i.e., a direct sum of copies of A.

(iv) The functor $Hom_A(P, ?)$ on the category of A-modules is exact.

Proof. To prove that (i) implies (ii), take $N = P$. To prove that (ii) implies (iii), take M to be free (this can always be done since any module is a quotient of a free module). To prove that (iii) implies (iv), note that the functor $\mathrm{Hom}_A(P, ?)$ is exact if P is free (as $\mathrm{Hom}_A(A, N) = N$), so the statement follows, as if the direct sum of two complexes is exact, then each of them is exact. To prove that (iv) implies (i), let K be the kernel of the map α, and apply the exact functor $\mathrm{Hom}_A(P, ?)$ to the exact sequence

$$0 \to K \to M \to N \to 0.$$

□

Definition 7.2. A module satisfying any of the conditions (i)-(iv) of Theorem 7.1 is said to be **projective**.

7.2 Lifting of idempotents

Let A be a ring, and $I \subset A$ a nilpotent ideal.

Proposition 7.3. *Let $e_0 \in A/I$ be an idempotent, i.e., $e_0^2 = e_0$. There exists an idempotent $e \in A$ which is a lift of e_0 (i.e., it projects to e_0 under the reduction modulo I). This idempotent is unique up to conjugation by an element of $1 + I$.*

Proof. Let us first establish the statement in the case when $I^2 = 0$. Note that in this case I is a left and right module over A/I. Let e_* be any lift of e_0 to A. Then $e_*^2 - e_* = a \in I$, and $e_0 a = a e_0$. We look for e in the form $e = e_* + b$, $b \in I$. The equation for b is $e_0 b + b e_0 - b = a$.

Set $b = (2e_0 - 1)a$. Then

$$e_0 b + b e_0 - b = 2e_0 a - (2e_0 - 1)a = a,$$

so e is an idempotent. To classify other solutions, set $e' = e + c$. For e' to be an idempotent, we must have $ec + ce - c = 0$. This is equivalent to saying that $ece = 0$ and $(1 - e)c(1 - e) = 0$, so $c = ec(1 - e) + (1 - e)ce = [e, [e, c]]$. Hence $e' = (1 + [c, e])e(1 + [c, e])^{-1}$.

Now, in the general case, we prove by induction in k that there exists a lift e_k of e_0 to A/I^{k+1}, and it is unique up to conjugation by an element of $1+I^k$ (this is sufficient as I is nilpotent). Assume it is true for $k=m-1$, and let us prove it for $k=m$. So we have an idempotent $e_{m-1} \in A/I^m$, and we have to lift it to A/I^{m+1}. But $(I^m)^2 = 0$ in A/I^{m+1}, so we are done. □

Definition 7.4. A complete system of orthogonal idempotents in a unital algebra B is a collection of elements $e_1, ..., e_n \in B$ such that $e_i e_j = \delta_{ij} e_i$, and $\sum_{i=1}^n e_i = 1$.

Corollary 7.5. *Let $e_{01}, ..., e_{0m}$ be a complete system of orthogonal idempotents in A/I. Then there exists a complete system of orthogonal idempotents $e_1, ..., e_m$ ($e_i e_j = \delta_{ij} e_i$, $\sum e_i = 1$) in A which lifts $e_{01}, ..., e_{0m}$.*

Proof. The proof is by induction in m. For $m = 2$ this follows from Proposition 7.3. For $m > 2$, we lift e_{01} to e_1 using Proposition 7.3, and then apply the induction assumption to the algebra $(1-e_1)A(1-e_1)$. □

7.3 Projective covers

Obviously, every finitely generated projective module over a finite dimensional algebra A is a direct sum of indecomposable projective modules, so to understand finitely generated projective modules over A, it suffices to classify indecomposable ones.

Let A be a finite dimensional algebra, with simple modules $M_1, ..., M_n$.

Theorem 7.6. *(i) For each $i = 1, ..., n$ there exists a unique indecomposable finitely generated projective module P_i such that $\dim Hom(P_i, M_j) = \delta_{ij}$.*

(ii) $A = \oplus_{i=1}^n (\dim M_i) P_i$.

(iii) any indecomposable finitely generated projective module over A is isomorphic to P_i for some i.

Proof. Recall that $A/\mathrm{Rad}(A) = \oplus_{i=1}^n \mathrm{End}(M_i)$, and $\mathrm{Rad}(A)$ is a nilpotent ideal. Pick a basis of M_i, and let $e_{ij}^0 = E_{jj}^i$, the rank 1 projectors projecting to the basis vectors of this basis ($j = 1, ..., \dim M_i$). Then e_{ij}^0 are orthogonal idempotents in $A/\mathrm{Rad}(A)$. So by Corollary 7.5 we can lift them to orthogonal idempotents e_{ij} in A. Now define $P_{ij} = Ae_{ij}$. Then $A = \oplus_i \oplus_{j=1}^{\dim M_i} P_{ij}$, so P_{ij} are projective. Also, we have $\mathrm{Hom}(P_{ij}, M_k) = e_{ij} M_k$, so $\dim \mathrm{Hom}(P_{ij}, M_k) = \delta_{ik}$. Finally, P_{ij} is independent of j up to an isomorphism, as e_{ij} for fixed i are conjugate under $A^\times$ by Proposition 7.3; thus we will denote P_{ij} by P_i.

We claim that P_i is indecomposable. Indeed, if $P_i = Q_1 \oplus Q_2$, then $\mathrm{Hom}(Q_l, M_j) = 0$ for all j either for $l = 1$ or for $l = 2$, so either $Q_1 = 0$ or $Q_2 = 0$.

Also, there can be no other indecomposable finitely generated projective modules, since any such module has to occur in the decomposition of A. The theorem is proved. □

References

[BGP] J. Bernstein, I. Gelfand, V. Ponomarev, Coxeter functors and Gabriel's theorem, Russian Math. Surveys 28 (1973), no. 2, 17–32.

[Cu] C. Curtis, Pioneers of Representation Theory: Frobenius, Burnside, Schur, and Brauer, AMS, 1999.

[CR] C. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, AMS, 2006.

[FH] W. Fulton and J. Harris, Representation Theory, A first course, Springer, New York, 1991.

[Fr] Peter J. Freyd, Abelian Categories, an Introduction to the Theory of Functors. Harper and Row (1964).

[McL] S. MacLane, Categories for a working Mathematician: 2nd Ed., Graduate Texts in Mathematics 5, Springer, 1998.