18.725: EXERCISE SET 10

DUE TUESDAY DECEMBER 2

(1) Find the singular points of the following curves in $\mathbb{A}^2$

(i) $V(x^2 - x^4 - y^4)$,

(ii) $V(x^2y + xy^2 - x^4 - y^4)$.

(2) Show that if X is a variety, then the set of smooth points of X is an open set.

(3) Let C be a complete smooth curve. Show that any non–constant $f \in k(X)$ defines a surjection $\pi_f : C \to \mathbb{P}^1$ such that for any point $p \in \mathbb{P}^1$ the inverse image $\pi_f^{-1}(f)$ is finite.

(4) Let $\overline{C}$ be a complete curve (not necessarily smooth). Let C be the complete smooth curve associated to the function field $k_{\overline{C}}$. Show that there is a natural surjective morphism $C \to \overline{C}$.

(5) The purpose of this exercise is to classify all automorphisms of $\mathbb{P}^1$. Let PGL_1 denote the quotient of $GL_2(k)$ (2×2 invertible matrices with entries in k) by the normal subgroup $k^* \subset GL_2(k)$ consisting of the diagonal matrices. Your task in this exercise is to show that the group of automorphisms of $\mathbb{P}^1$ is equal to PGL_1.

To do so, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(k)$ be a matrix and define a map

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^1, \quad [x : y] \mapsto [ax + by, cx + dy].$$

Show that this is a well–defined automorphism. Then show that the induced map $GL_2(k) \to \mathrm{Aut}(\mathbb{P}^1)$ induces an isomorphism of groups $PGL_1 \simeq \mathrm{Aut}(\mathbb{P}^1)$. Hint: classify the k–automorphisms of $k(\mathbb{P}^1) \simeq k(t)$.

(6) Let X be an affine variety, and let $\mathrm{Op}(X)$ be the collection of open subsets of X viewed as a category as in class. Let $\mathrm{SOp}(X) \subset \mathrm{Op}(X)$ be the subcategory of special opens. In other words, the objects of $\mathrm{SOp}(X)$ are open sets $D(f)$ for $f \in \Gamma(X, \mathcal{O}_X)$ and a morphism $D(f) \to D(g)$ is just an inclusion. Any sheaf $F : \mathrm{Op}(X) \to (\mathrm{Set})$ defines a functor $\bar{F} : \mathrm{SOp}(X) \to (\mathrm{Set})$ simply by restricting the functor. Prove the converse. That is, let $\bar{F}$ be a functor

$$\bar{F} : \mathrm{SOp}(X) \longrightarrow (\mathrm{Set})$$

such that if $D(f) = \cup_i D(f_i)$ the sequence

$$\bar{F}(D(f)) \to \prod_i \bar{F}(D(f_i)) \rightrightarrows \prod_{i,j} \bar{F}(D(f_i f_j))$$

is exact. Then show that there exists a unique sheaf F on X whose restriction to $\mathrm{SOp}(X)$ is $\bar{F}$.

(7) Let X be a variety. A sheaf of $\mathcal{O}_X$–modules $\mathcal{L}$ is called an *invertible sheaf* if there exists an open covering $X = \cup U_i$ such that $\mathcal{L}$ restricted to each U_i is isomorphic to $\mathcal{O}_X$ viewed as a

Date: November 21, 2003.

module over itself. If $\mathcal{L}$ and $\mathcal{M}$ are two invertible sheaves, let $\mathcal{L} \otimes \mathcal{M}$ be the sheaf associated to the presheaf which to any open U associates the tensor product $\mathcal{L}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{M}(U)$. Show that $\mathcal{L} \otimes \mathcal{M}$ is again an invertible sheaf, and that the operation $(\mathcal{L}, \mathcal{M}) \mapsto \mathcal{L} \otimes \mathcal{M}$ makes the set of isomorphism classes of invertible sheaves into an abelian group.