18.725: FINAL EXAM

DUE WEDNESDAY DECEMBER 10

(1) Show that any complete smooth curve C of genus 2 admits a finite morphism $f : C \to \mathbb{P}^1$ such that the degree of the field extension $k(\mathbb{P}^1) \to k(C)$ is 2. Hint: Look at Ω^1_C.

(2) Let C be a smooth complete curve and let $P_1, \dots, P_r \in C$ be some points. Set $U = C - \{P_1, \dots, P_r\}$. Show that there exists a non–constant $f \in \Gamma(U, \mathcal{O}_C)$.

(3) (i) Let C be a smooth curve which is not complete. Let $\overline{C}$ be the complete smooth curve associated to the field of rational functions $k(C)$. Show that C is naturally identified with an open subset of $\overline{C}$.

(ii) Combining (2) with (i), show that C is affine. Hint: Use (2) to show that there exists a finite morphism $g : \overline{C} \to \mathbb{P}^1$ such that $g^{-1}(\mathbb{A}^1) = C$. Then show that C is affine with coordinate ring equal to the normalization of $k[x]$ in $k(C)$.

(4) Let $f(x) \in k[x]$ be a polynomial of degree $d \geq 1$ with no multiple roots. Let $C \subset \mathbb{A}^2$ be the affine curve defined by

$$y^2 = f(x).$$

Let $\overline{C}$ be the closure of the image of the map

$$C \longrightarrow \mathbb{P}^{g+2}, \quad [x : y] \mapsto [1 : x : x^2 : \cdots : x^{g+1} : y].$$

(i) Show that $\overline{C}$ is a complete smooth curve and that $\overline{C} \cap U_{X_0 \neq 0} \simeq C$.

(ii) Write $f(x) = a_0x^d + a_1x^{d-1} + \cdots + a_d$ and let $g(v) = a_0 + a_1v + \cdots + a_{d-1}v^{d-1} + a_dv^d$. Show that $\overline{C}$ has an open cover consisting of the two curves

$$V(y^2 - f(x)) \subset \mathbb{A}^2, \quad V(w^2 - g(v)) \subset \mathbb{A}^2.$$

(iii) Let g be the unique integer satisfying $d - 3 < 2g \leq d - 1$. Show that the curve $\overline{C}$ has genus g.

(5) Extra Credit: You might guess that $\mathbb{P}^n$ should represent the functor

$$F : (\text{varieties}) \longrightarrow (\text{Set})$$

which to any variety X associates the set of rank 1 free $\Gamma(X, \mathcal{O}_X)$–submodules $L \subset \Gamma(X, \mathcal{O}_X)^{n+1}$. Prove that this guess is incorrect by showing that the above functor F is not representable. Hint: morphisms between varieties can be glued.

Date: December 4, 2003.