18.725: EXERCISE SET 4

DUE THURSDAY OCTOBER 9

(1) Let X be an affine variety, and suppose $Z \subset X$ is a closed subset. Let $\mathcal{I} \subset \mathcal{O}_X$ be the presheaf which to each $U \subset X$ associates the set of elements $f \in \mathcal{O}_X(U)$ for which $f(x) = 0$ for every $x \in U \cap Z$. Show that $\mathcal{I}$ is a sheaf of ideals in $\mathcal{O}_X$.

(2) (i) Let $Z \subset \mathbb{A}^2$ be the closed subset defined by $xy = 1$. Is Z irreducible?

(ii) What about $E \subset \mathbb{A}^2$ defined by $y^2 = x(x-1)(x-\lambda)$, where $\lambda \in k$.

(3) Let X be an affine variety, and let $Y \subset \mathbb{A}^n$ be an affine variety. Let $x_i \in \Gamma(Y, \mathcal{O}_Y)$ be the function which sends $(a_1, \ldots, a_n) \in Y$ to a_i. Show that a continuous map $f : X \to Y$ is a morphism if and only if for each $i = 1, \ldots, n$ the composite function $x_i \circ f : X \to k$ is in $\Gamma(X, \mathcal{O}_X)$.

(4) Let X be an affine variety such that $\Gamma(X, \mathcal{O}_X)$ is a unique factorization domain. Show that if $f \in \Gamma(U, \mathcal{O}_X)$ for some open $U \subset X$, then there exists $p, q \in \Gamma(X, \mathcal{O}_X)$ such that $q(a) \neq 0$ for each $a \in U$ and $f(a) = p(a)/q(a)$.

(5) Let $D = \{x^2 + y^2 < 1\}$ be the unit disc in $\mathbb{R}^2$ with the topology induced by the standard topology on $\mathbb{R}^2$. Let F be the sheaf on D which to any $U \subset D$ associates the set of differentiable real–valued functions $U \to \mathbb{R}$. Show that the stalk $F_{(0,0)}$ of F at $(0,0)$ is a local ring. Hint: show that there is a natural surjection $F_{(0,0)} \to \mathbb{R}$.

(6) Let X be a topological space, and $\{U_i\}$ an open covering of X. Suppose given a collection $\{\mathcal{F}_i\}$ of sheaves $\mathcal{F}_i$ on U_i, together with isomorphisms

$$\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j} \quad (0.0.0.1)$$

such that

1. for each i, φ_{ii} is the identity;
2. for each triple i, j, k, the map φ_{ik} is equal to $\varphi_{jk} \circ \varphi_{ij}$.

Show that there exists a unique sheaf $\mathcal{F}$ on X together with isomorphisms $\psi_i : \mathcal{F}|_{U_i} \to \mathcal{F}_i$ such tat for each i and j, the maps ψ_j and $\varphi_{ij} \circ \psi_i$ over $U_i \cap U_j$ are equal.

(7) A prevariety X is called *rational* if its function field $k(X)$ is isomorphic to the field of fractions $k(x_1, \ldots, x_n)$ of a polynomial ring $k[x_1, \ldots, x_n]$ for some n. Let $X \subset \mathbb{A}^4$ be the hypersurface defined by $wx = yz$. Show that X is rational.

(8) Compute $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})$.

Date: October 2, 2003.