18.725: EXERCISE SET 5

DUE THURSDAY OCTOBER 16

(1) Let X and Y be prevarieties. Define h_Y to be the presheaf on X which to any open set $U \subset X$ associates the set of morphisms of prevarieties $U \to Y$. Show that h_Y is a sheaf. Also show that when $Y = \mathbb{A}^1$, then h_Y is just $\mathcal{O}_X$ viewed as a sheaf of sets.

(2) Let k be an algebraically closed field, K and K' field extensions of k. Show that $K \otimes_k K'$ is again a field. Give an example to show that this is false if k is not algebraically closed.

(3) Fix an invertible 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Define a map $\mathbb{P}^1 \to \mathbb{P}^1$ by sending $[x : z]$ to $[ax + bz : cx + dz]$. Show that this is an automorphism of $\mathbb{P}^1$ (i.e. a morphism which is an isomorphism).

(4) Show that any two projective varieties in $\mathbb{P}^2$ defined by a homogeneous equation of degree 2 are isomorphic. Hint: look up classification of quadratic forms for example in Lang's *Algebra*.

(5) Let $E \subset \mathbb{A}^2$ be the affine variety defined by $y^2 - x^3 - x = 0$. Show that for any point $P \in E$, the set $E - P$ is again an affine variety.

(6) (i) Show that the map $\pi : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ sending $(x_0, \ldots, x_n)$ to $[x_0 : \cdots : x_n]$ is a morphism of prevarieties.

(ii) If X is any other prevariety, show that a continuous map $f : \mathbb{P}^n \to X$ is a morphism if and only if the composite $f \circ \pi : \mathbb{A}^{n+1} - \{0\} \to X$ is a morphism.

(7) Explain in detail what it means to say that the product of two prevarieties is unique.

Correction to exercise 2:

Show that $K \otimes_k K'$ is an integral domain and that this is false if 'k' is not algebraically closed.

Date: October 9, 2003.