18.725: EXERCISE SET 6

DUE THURSDAY OCTOBER 23

(1) Suppose X is an affine variety, and let $\Delta : X \to X \times X$ be the diagonal morphism. Describe explicitly the ring map

$$\Gamma(X \times X, \mathcal{O}_{X\times X}) \longrightarrow \Gamma(X, \mathcal{O}_X).$$

(2) Let X, Y, Z be three prevarieties. Show that there is a natural isomorphism (where the parentheses indicate in which order you do the products

$$(X \times Y) \times Z \simeq X \times (Y \times Z).$$

(3) Let X be the affine line with the origin doubled discussed in class (i.e. two copies of $\mathbb{A}^1$ glued along $\mathbb{A}^1 - \{0\}$). Describe explicitly the product $X \times X$ and show directly that the diagonal $\Delta(X) \subset X \times X$ is not closed.

(4) Fix a positive integer d, and let $M_0, \ldots, M_N \in k[X_0, \ldots, X_n]$ be all monomials of degree d. The *Veronese embedding* is the morphism $V_d : \mathbb{P}^n \to \mathbb{P}^N$ defined by

$$V_d(x_0 : \cdots : x_n) = (M_0(\underline{x}) : \cdots : M_N(\underline{x})).$$

(a) Show that V_d is an isomorphism of $\mathbb{P}^n$ with a closed projective variety in $\mathbb{P}^N$.

(b) Let $S \subset \mathbb{P}^n$ be a hypersurface defined by a homogeneous polynomial $f \in k[X_0, \ldots, X_n]$ of degree d. Show that $S = V_d^{-1}(H)$ for a unique hyperplane $H \subset \mathbb{P}^N$.

(5) A *group pre-variety* is a prevariety G together with morphisms $m : G\times G \to G$, $i : G \to G$, and an identity element $e \in G$ such that the underlying set is a group with group law defined by m, i, and e. Show that any group pre-variety is a variety.

(6) A pre–variety X is *quasi–affine* if it is isomorphic to an open subset of an affine variety. If X is a prevariety and $f \in \Gamma(X, \mathcal{O}_X)$, let X_f be the open set $\{x \in X | f(x) \neq 0\}$. Prove that a pre–variety is quasi–affine if and only if the sets X_f for $f \in \Gamma(X, \mathcal{O}_X)$ form a base for the topology of X.

(7) (Yoneda's lemma). Let $\mathcal{C}$ be a category. For any object $X \in \mathrm{Ob}(\mathcal{C})$, let

$$h_X : \mathcal{C} \longrightarrow (\mathrm{Set})$$

be the functor which to any $Y \in \mathcal{C}$ associates $\mathrm{Hom}_{\mathcal{C}}(Y, X)$. Convince yourself (and the grader) that h_X is a functor. Then show that if $Y \in \mathcal{C}$ is a second object, there is a natural bijection

$$\mathrm{Hom}_{\mathcal{C}}(Y, X) \simeq \mathrm{Hom}(h_Y, h_X),$$

where $\mathrm{Hom}(h_Y, h_X)$ is the set of natural transformations of functors $h_Y \to h_X$.

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As an illustration, consider $\mathbb{A}^n$ and let $f_1, \ldots, f_n \in k[X_0, \ldots, X_n]$ be some polynomials. Taking $\mathcal{C}$ in the above to be the category of pre–varieties, describe the functor $h_{\mathbb{A}^n}$ and the morphism of functors $h_{\mathbb{A}^n} \to h_{\mathbb{A}^n}$ giving rise to the morphism of varieties

$$\mathbb{A}^n \longrightarrow \mathbb{A}^n, \quad (x_1, \ldots, x_n) \mapsto (f_1(\underline{x}), \ldots, f_n(\underline{x})).$$