18.725: EXERCISE SET 7

DUE THURSDAY OCTOBER 30

(1) Show that if X is a variety, then the intersection of any two affine opens in X is again affine. Give an example to show that this is false for a general pre–variety.

(2) Let $X \subset \mathbb{P}^n$ be a projective variety with homogeneous ideal $I(X) \subset k[X_0, \ldots, X_n]$. Let $X^* \subset \mathbb{A}^{n+1}$ denote the affine variety defined by $I(X)$. Show that the dimension of X^* is one more than the dimension of X.

(3) Let X be a variety of dimension n. Show that any sequence of irreducible closed subsets

$$\emptyset \subsetneq Z_0 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_r = X$$

can be completed to a sequence

$$\emptyset \subsetneq Z'_0 \subsetneq Z'_2 \subsetneq \cdots \subsetneq Z'_n = X$$

of length n. In other words, such that for each $i \in [1, r]$ there exists a $j \in [1, n]$ such that $Z_i = Z'_j$.

(4) Let X be a variety of dimension 1, and suppose $\pi : X \to Y$ is a non–constant morphism to a variety Y. Show that the fiber of π are finite.

(5) Let $\pi : X \to \mathbb{A}^n$ be the blow-up of $\mathbb{A}^n$ at the point $(0, \ldots, 0)$ discussed in class. Show that X is not an affine variety.

(6) Let $Z \subset \mathbb{A}^2$ be the algebraic set defined by $xy = 0$. Let $X \subset \mathbb{A}^2 \times \mathbb{P}^1$ be the blow–up of $\mathbb{A}^2$ at $(0, 0)$. Describe $\pi^{-1}(Z)$, where $\pi : X \to \mathbb{A}^2$ is the blow–up map. Also describe the closure in X of the set $\pi^{-1}(Z - \{(0, 0)\})$.

(7) (An example of why we need schemes). Let C be a category and

$$p_1 : X \to Z, \quad p_2 : Y \to Z$$

two morphisms in C. A *fiber product* of X and Y over Z is a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\pi_2} & Y \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ X & \xrightarrow{p_1} & Z \end{array}$$

which is the universal such commutative diagram. In other words, given any other object $F \in C$ and morphisms $\rho_1 : F \to X$ and $\rho_2 : F \to Y$ such that $p_1 \circ \rho_1 = p_2 \circ \rho_2$, there exists a unique morphism $\lambda : F \to W$ such that $\rho_i = \pi_i \circ \lambda$.

Show that fiber products do not exist in the category of affine varieties. Hint: try to construct the fiber product using the method we used to construct products and see what goes wrong.

Date: October 23, 2003.