September 30, 2003

1 Review of things not covered enough

Topics: fibers, morphisms of sheaves.

Def. A *fiber* of a morphism is the inverse image of a point.

Recall. A sheaf is a functor $F : Op(X) \to Set, Ab, Group$, et cetera. Think of it this way: F is a rule that associates to every open $U \subset X$ a set $F(U)$ and for every inclusion $V \subset U$, there is a restriction map $res_{UV} F(U) \to F(V)$, satisfying [... the rules of a functor.]

Suppose we have $F, G : Op(X) \to Set$. A morphism $\gamma : F \to G$ of sheaves is a "natural transformation of functors." This means that for every $U \subset X$ open we have a map $\gamma_U : F(U) \to G(U)$, such that for every $V \subset U$, the diagram

$$\begin{array}{ccc} F(U) & \xrightarrow{\gamma_U} & G(U) \\ res^F_{UV} \downarrow & & \downarrow res^G_{UV} \\ F(V) & \xrightarrow{\gamma_V} & G(V) \end{array}$$

commutes.

1.1 Tensor products

The tensor product is defined by universal properties.

2 Back to work

Suppose we have X an affine irreducible algebraic set. We have shown that under the Zariski topology, there is a natural sheaf of rings O_X. We can think of this as a sheaf of certain functions on X (i.e. $U \mapsto$ special maps $U \to k$. Also, we can think of $O_X(U) = \cap_{x \in U} R_{m_x} \subset K$ where $R = \Gamma(X)$ and $K = \text{Frac } (R)$.

Today, think of $X \subset k^n$. We have a quotient map $k[x_1, \ldots, x_n] \twoheadrightarrow R$. R_{m_x} is a localization; $h \in R_{m_x}$ is of the form $h = f/g$ where $f, g \in R$ and $g \notin m_x$. Thus, there exist polynomials $\tilde{f}, \tilde{g} \in k[x_1, \ldots, x_n]$ inducing f, g, and we can let $\tilde{h} = \tilde{f}/\tilde{g}$. We can evaluate this whenever the denominator is nonzero. Basically, $O_X(U)$ consists of all such $\tilde{f}/\tilde{g}$ for which $\tilde{g}$ is nonzero everywhere on U, so we can evaluate to get an element of k.

If $f \in R$ and $f(a_1, \ldots, a_n) = 0$ for every $\underline{a} \in X$ then $V(f) \supset X$ in k^n and so $I(X) \supset (f)$, so $f = 0$ in R.

Now, the map $R_{m_x} \to k$ defined by $h \mapsto h(x)$ is just the quotient map $R_{m_x} \twoheadrightarrow R_{m_x}/m_x = k$, which is the evaluation map at the point x associated to m_x.

3 Morphisms

We have four equivalent things that define a morphism (from last time). The following are equivalent for a continuous map $f : X \to Y$ where X and Y are irreducible affine alg. sets.

(1) f is a morphism

(2) For all $U \subset Y$, maps $g : U \to k$ in $O_Y(U)$, the composite map $f^{-1}(U) \xrightarrow{f} U \xrightarrow{g} k$ is in $O_X(f^{-1}(U))$.

(3,4) [we don't need these now, so don't worry about them.]

To restate property (2), the map f induces a morphism of sheaves $O_Y \to f_*O_X$. To check that this is the same, we need that for every $U \subset Y$, there is a map $O_Y(U) \to (f_*O_X)(U) = O_X(f^{-1}(U))$. The map is simply $g \mapsto g \circ f$. We have to check this works with respect to restrictions. Suppose we have $V \subset U$. We want to check that the diagram commutes. So we need to check that $(res_{UV}g) \circ f = res_{UV}(g \circ f)$. The RHS is just $g \circ f|_{f^{-1}(V)}$. The LHS is $g|_V \circ f|_{f^{-1}(V)}$ (this is sloppy notation; actually it's a different restriction), and obviously these are the same; we don't need to restrict g to V when we've already restricted to points that lead to points in V.

4 Varieties

Def. An *affine variety* is a pair (X, O_X) where X is a topological space, and O_X is a sheaf of k-valued functions, which is isomorphic to the topological space with sheaf of functions coming from an irreducible affine algebraic set.

Def. A *morphism of varieties* $(X, O_X) \to (Y, O_Y)$ is a continuous map $f : X \to Y$ so that composition with f induces a morphism $O_Y \to f_*O_X$.

This is nothing new; everything we had before is a variety and everything that was a morphism is a morphism here. Now, though, we can see how to globalize.

Def. A *prevariety* is a pair (X, O_X) where X is a toplogical space, O_X is a sheaf of k-valued functions on X, such that (i) X is connected, and (ii) $\exists$ a finite open cover $X = \cup_{i=1}^n U_i$ such that $(U_i, O_X|_{U_i})$ is an affine variety.

A variety will be a prevariety that satisfies some additional conditions.

Ex. Say that the characteristic of $k = p > 0$, $f : \mathbb{A}^1 \to \mathbb{A}^1$, defined by $f_* : k[t] \to k[t]$ that maps $t \mapsto t^p$. Look at this as a map of topological spaces $\mathbb{A}^1 \to \mathbb{A}^1$. This is a homeomorphism (a special case of problem 6 on hw 3), but not an isomorphism! The inverse map doesn't give a map of rings because taking a pth root is not a polynomial thing.

Ex. Consider $\mathbb{A}^1 \to \mathbb{A}^2$ defined by $t \mapsto (t^2, t^3)$. This is a bijection. In fact, it is a homeomorphism. The associated map $k[X, Y]/(X^3 - Y^2) \to k[t]$ defined by $X \mapsto t^2$ and $Y \mapsto t^3$ is not an isomorphism, so this is not an isomorphism of varieties.

Ex. Consider $\mathbb{A}^1 \to X$ where X is defined by $Y^2 = X^2(X+1)$, the map being defined by $t \mapsto (t^2 - 1, t(t^2 - 1))$. The map $k[X, Y]/(Y^2 - X^3 - X^2) \to k[t]$ is again not an isomorphism despite that the map from $\mathbb{A}^1 \to X$ is a bijection. Here, ± 1 map to the same point, and we can think of the sheaf of functions on X as functions on $\mathbb{A}^1$ that agree at ± 1.

From the homework, $\mathbb{A}^2 - \{(0, 0)\}$ is a prevariety but not a variety (it is easy to cover that space with open sets).

Lemma: Say $U \subset X$ is equal to $D(f)$ for some $f \in R = \Gamma(X, O_X)$. Then, $(U, O_X|_U)$ is an affine variety.

Proof $X \subset k^n$ where $k[x_1, \ldots, x_n] \twoheadrightarrow R$. Let $f \in R$, and corresponding $\tilde{f}$ in $k[x_1, \ldots, x_n]$. Consider $k[x_1, \ldots, x_{n+1}]/(I, x_{n+1}\tilde{f} - 1)$. Call the affine variety associated to this Σ. We have $\Sigma \to X$ defined by restriction to the first n coordinates. This maps bijectively to U. This is a homeomorphism (check). Given that, $(\Sigma, O_\Sigma) \to (U, O_X|_U)$ makes sense... think of $O_X|_U$ and O_Σ as being sheaves on the same topological space, and consider the natural map $O_X|_U \to O_\Sigma$ mapping $O_X(V) \to O_\Sigma(V)$ for any $V \subset U$ open. This is an isomorphism of sheaves, which proves that $(U, O_X|_U)$ is an affine variety as it is isomorphic to (Σ, O_Σ) which is clearly an affine variety. [Some confusing details omitted]

Lemma: Let $Z \subset X$ be closed. Then (Z, O_Z) is an affine variety. This is pretty obvious since closed subsets of affine algebraic sets are affine algebraic sets. The only question is what O_Z is, but this is just $O_X|_Z$. The proof Martin gave in class is much more confusing.