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Remark. If X is a prevariety, then O_X is a sheaf of rings of k-valued functions. That is, elements of O_X (like elements of O_X when X is a variety) can be evaluated on X and can be added and multiplied. Proof is by reduction to the finite affine open cover of each open set.

Remark. If X is a prevariety, the function field $k(X) = \overset{\lim}{\rightarrow}_U \Gamma(U, O_X)$.
So, given any $U \subset X$ which is open, we can mape $\Gamma(U, O_X) \to k(X)$ where $f \mapsto \overline{f}$.

Claim. This map is injective. Suppose we have f, g which both map to $\overline{f}$. This is certainly true for affine open sets U, so it's true in general by the finite cover, because they must be the same when restricted to *some* affine open set, because they are the same in the limit.

Prop. X is a prevariety. Then $U \subset X$ is open implies that $(U, O_X|_U)$ is a prevariety.

Pf. To show that U is connected, if $U = U_1 \cup U_2$ and $U_1 \cap U_2 = \emptyset$ then U_1 and U_2 are open sets in X which don't meet, which is a contradiction. To show it has a finite affine open cover, just take the affine open cover V_i and intersect each of these with U. Each of these is a prevariety by the lemma we proved before (that if $U \subset X$ is an *affine* open set, then U makes a prevariety), so each has a finite affine open cover, so we take all those sets together to cover U.

Example. $\mathbb{A}^2 - \{(0,0)\}$ is not a variety, but has a natural structure of a prevariety. The affine open covering is $\mathbb{A}^2 - \{(x,0)\}$ and $\mathbb{A}^2 - \{0, y\}$. We proved it is not a variety on HW.

Let X be a prevariety, $Z \subset X$ be an irreducible closed set.

Then $O_Z(V) = \{f : Z \supset V \to k | \forall x$ there is an open neighborhood U of x in X such that f is the restriction of an element of $O_X(U)\}$. Our claim is that (Z, O_Z) is a prevariety.

Pf. This is true on an affine variety X. However, $X = \cup V_i$ where V_i are affine open sets, and $Z \cap V_i$ are prevarieties: the definition of $O_Z(V)$ for X affine is the same as this. So we just put 'em together again.

Def. T is a topological space, $R \subset T$ is *locally closed* if $\exists$ an open set $U \subset T$ such that $R \subset U$ and is closed in U. For example, $\mathbb{A}^2 - \{(x,0) | x \neq 0\}$ is neither open nor closed. However, in $D(x)$ this is closed.

Remark. If X is a prevariety, $R \subset X$ is closed, then R has a natrual structure of a prevariety.

1 Projective Varieties

That is, making $\mathbb{P}^n$ a prevariety. We already did $\mathbb{P}^1 = U_{x \neq 0} \cup U_{y \neq 0}$ We map $(a, b) \to (b/a)$ in $U_{x \neq 0}$ and to (a/b) in $U_{y \neq 0}$. So $(y/x) \mapsto (x/y)$ and vice versa, on the intersection $U_{x \neq 0} \cap U_{y \neq 0}$.

Say that $R = \oplus_{n=0}^{\infty} R_n$ is a graded ring (ie, polynomial rings where R_i is all polynomials of a certain degree) and an integral domain.

We say that $\rho \subset R$ is a homogenous prime ideal. We get R_ρ, the localization, is also a graded ring. Write $(R_\rho)_0$ for the degree 0 part of R_ρ. This matches up with what we want: for instance, $(k[x,y]_x)_0 = k[(y/x)]$: everything in $k[(y/x)]$ has degree 0. So $(R_\rho)_0 = \{f/g | f, g \in R_n, g \in R - \rho\}$ We want: $X \subset \mathbb{P}^n$ irreducible $\iff$ $V(P) : P \subset k[x_0, \ldots, x_n]$ is a homogenous prime. So then $O_X(U) = \cap_{x \in U} O_{X,x} \subset k(X)$. Write $R = k[x_0, \ldots, x_n]/P$. Note that this makes sense as a graded ring!

$k(X) = (\text{Frac }(R))_0$. Let $f/g \in k(X)$, $f, g \in R_n$, so we can say that $g(\underline{x}) \neq 0$ for $\underline{x} \in X$. Let $O_{X,\underline{x}} = \{f/g | f, g R_n, g(\underline{x}) \neq 0\} \subset k(X)$.

Remark. $O_{X,\underline{x}}$ is a local ring. This is shown by the evaluation map $f/g \mapsto (f/g)(\underline{x})$; the kernel is the set $\{f/g | f(\underline{x}) = 0\}$

Claim. (X, O_X) is a prevariety. We need $(U_i \cap X, O_X|_{U_i \cap X})$ is an affine variety where $\cup U_i = \mathbb{P}^n$ where U_i is the standard covering. We also need to check that $k(X)$ really is the function field. We know $(U_i \cap X, A)$ is an affine variety (where A is whatever makes it an affine variety. We want to show that A and O_{U_i} match up. In order to check this, we only need to check that $k(X)$ is the same function field, and then it will all work out. It does [mess ommitted].

Def. A *morphism* of prevarieties $f : X \to Y$ is a continuous map such that for every open $V \subset Y$ and $g : V \to k$ in $O_Y(V)$, the composite $f^{-1}(V) \xrightarrow{f} V \xrightarrow{g} k$ is in $O_X(f^{-1}(V))$.

Example. $C \subset \mathbb{P}^2$ defined by $XZ = Y^2$. $\Sigma_2 = \mathbb{P}^1$, $C \cong \mathbb{P}^1$. Fix $P_0 = [0 : 0 : 1]$ and $Q_0 = [1 : 0 : 0]$, then the morphism is given by $C - P_0 : [a : b : c] \mapsto [a : b]$ and $C - Q_0 : [a : b : c] \mapsto [b : c]$.

Prop. Say $f : X \to Y$ is a continous map. Then

1. If $\exists$ an open cover V_i of Y such that $f^{-1}(V_i) \xrightarrow{f} V_i$ is a morphism, then f is a morphism.
2. If $\exists$ an open cover U_i of X such that $U_i \mapsto Y$ is a morphism, then f is a morphism.

We will prove this proposition next time. For now, let's use it to prove that the morphism in our example is an isomorphism.

$$\begin{array}{ccc} C - P_0 = C \cap U_{x \neq 0} & \to & U_{s \neq 0} \\ \updownarrow & & \updownarrow \\ k[(y/x), (z/x)]/((y/x)^2 - (z/x)) & \leftarrow (\cong) & k[(t/s)] \\ (y/x) & \leftarrow & t/s, \end{array}$$

and similarly

$$\begin{array}{ccc} C - Q_0 = C \cap U_{z \neq 0} & \to & U_{t \neq 0} \\ \updownarrow & & \updownarrow \\ k[(x/z), (y/z)]/((y/z)^2 - (x/z)) & \leftarrow (\cong) & k[(t/s)] \\ (y/z) & \leftarrow & s/t. \end{array}.$$

On the overlap, $U_{x\neq 0} \cap U_{z\neq 0} \cap C = C \cap U_{y\neq 0}$, we have $k[(y/x),(z/x)^{\pm}]/((y/x)^2 - (z/x)) \cong k[(y/x)^{\pm},(z/x)^{\pm}]/((y/x)^2 - (z/x)) \cong k[(x/z)^{\pm},(y/z)^{\pm}]/((x/z - y/z)^2)$ defined by $(z/x)^{-1} \leftarrow (x/z)$.

$$\begin{array}{ccc} (z/x)^{-1} & \leftarrow & (x/z) \\ (y/x)(z/x)^{-1} & \leftarrow & (y/z) \\ k[(y/x)^{\pm},(z/x)^{\pm}]/((y/x)^2 - (z/x)) & \cong & k[(x/z)^{\pm},(y/z)^{\pm}]/((x/z) - (y/z)^2) \\ \cong\uparrow & & \cong\uparrow \\ k[(t/s)^{\pm}] & \cong\leftarrow & k[(s/t)^{\pm}]. \end{array}$$