1 Last time

We showed that $X \subset \mathbb{P}^n$ projective irreducible alg. set, then X has structure of a prevariety, st. each $U_i \cap X \subset X$ are open and affine.

Also, we can think of $I \subset k[x_0, \dots, x_n] \to R = \oplus_{n \geq 0} R_n$ a graded ring. Then $O_{X,x} = (R_{m_x})_0$. Similarly, for $U_{x_i \neq 0} \subset \mathbb{P}^n$, we get $O_X(U_{x_i \neq 0}) = (k[x_0, \dots, x_n]_{x_i})_0 = k[x_1/x_i, \dots, x_n/x_i]$.

We defined a morphism of prevarieties: if X, Y are prevarieties then a morphism $X \to Y$ is a continuous map $f : X \to Y$ such that $\forall V \subset Y$ open and $\forall g \in \Gamma(V, O_Y)$ the composite $g \circ f$ is in $\Gamma(f^{-1}(V), O_X)$.

Prop. Suppose X, Y are prevarieties, $f : X \to Y$ is continuous.

1. If there is a cover $Y = \cup V_i$ such that the restriction $f^{-1}(V_i) \to V_i$ is a morphism for every i then f is a morphism.
2. If there is a cover $X = \cup U_i$ such that $f|_{U_i} : U_i \to Y$ is a morphism for every i then f is a morphism.

Pf.
1. Let $V \subset Y$, $g \in \Gamma(V, O_Y)$. We have

$$\begin{array}{ccc} \Gamma(f^{-1}(V), O_X) & \to & \prod_i \Gamma(f^{-1}(V \cap V_i), O_X) & \rightrightarrows \prod_{i,j} \Gamma(f^{-1}(V \cap V_i \cap V_j), O_X) \\ & & (g \circ f|_{f^{-1}(V \cap V_i)}) \in & \end{array}$$

This inclusion is due to the assumption, and by exactness this means that $g \circ f$ is actually in $\Gamma(f^{-1}(V), O_X)$ so f is a morphism.

2. Same argument, but instead of using $f^{-1}(V \cap V_i)$ we use $f^{-1}(V) \cap U_i$.

2 Products of Varieties

Let $\mathcal{C}$ be a category, $X, Y \in Ob(\mathcal{C})$. Then a product of X and Y in $\mathcal{C}$ is a Z such that $Z \xrightarrow{p} X$ and $Z \xrightarrow{q} Y$ and such that if we have a map $W \xrightarrow{s} X$ and $W \xrightarrow{t} Y$ then there is a unique map $W \to Z$ such that all this commutes.

For example, $\mathcal{C} = Group$. Let G_1, G_2 be groups. Then the product of G_1 and G_2 is $G_1 \times G_2$ where multiplication is component-wise. If we have $H \xrightarrow{s} G_1$ and $H \xrightarrow{t} G_2$ then $H \xrightarrow{\rho} G_1 \times G_2$ defined by $\rho : h \mapsto (s(h), t(h))$.

Note: the direct sum is $G_1 \oplus G_2$; this has the complementary property; there are maps $G_1 \to G_1 \oplus G_2$ and $G_2 \to G_1 \oplus G_2$ such that if there are maps $G_1 \to H$ and $G_2 \to H$ then there is a unique map $G_1 \oplus G_2 \to H$ such that these all commute. In some instances, the direct sum and (direct) product are the same, but they are defined by these properties.

Another example: suppose we have an infinite set of vector spaces $\{V_i\}_{i=1}^{\infty}$, then there is an infinite product defined naturally: if there are maps from Z to each V_i then there is a unique map from Z to the infinite product so that these maps compose with the "restriction" maps from the product to each V_i. In this case, the direct product is simply infinite vectors where each component is from a particular V_i, while the direct sum is only such vectors where only finitely many are non-zero.

Theorem.

1. (Finite) products exist in the category of prevarieties.
2. If X and Y are affine varieties, then $X \times Y$ is an affine variety (in the product in the category of prevarieties: not what we proved on the homework).
3. If X and Y are projective varieties, then $X \times Y$ is a projective variety.

This may take some time to prove. Basically, we do this for the affine case, and then we do some gluing.

Remark. If it exists, as a set, $X \times Y$ is the product of the sets X and Y.

If X is a prevariety, then to give a point of X is the same as giving a morphism $(*, k) \to X$ where $(*, k)$ is the variety of one point. When we consider morphisms $(*, k) \to X \times Y$ then this must give a morphism $(*, k) \to X$ and $(*, k) \to Y$. We write $|X|$ to denote the set of X. Then in general, $|X| = \mathrm{Hom}((*, k)X)$. By the universal property,

$$|X \times Y| = \mathrm{Hom}((*, k), X \times Y) \cong \mathrm{Hom}((*, k), X) \times \mathrm{Hom}((*, k), Y) = |X| \times |Y|.$$

2.1 Affine case

Try to do the affine case. Let X, Y be affine. Let $R = \Gamma(X, O_X)$ and $S = \Gamma(Y, O_Y)$.

Lemma $R \otimes_k S$ is a finitely generated integral domain if R and S are. It is enough to consider R, S fields; we do this on HW5. It is enough because $R \otimes_k S$ can be thought of as sitting in Frac $(R) \otimes_k$ Frac (S); if this is an integral domain then $R \otimes_k S$ must be.

Remark. Define Z to be the affine variety associated to $R \otimes_k S$.

Lemma. Let X be a prevariety, Y an affine variety. Then the map

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(S, R)$$

is an isomorphism.

Remark. This proves that $Z = X \times Y$ is the product as prevarieties! If we have W a prevariety, $T = \Gamma(W, O_W)$,

$$\mathrm{Hom}(W, Z) \cong \mathrm{Hom}(R \otimes S, T) \cong \mathrm{Hom}(R, T) \times \mathrm{Hom}(S, T) \cong \mathrm{Hom}(W, X) \times \mathrm{Hom}(W, Y).$$

Pf. of lemma. Say $X = \cup_i U_i$ where U_i are affine open sets. Claim: the following is exact.

$$\mathrm{Hom}(X,Y) \to \prod_i \mathrm{Hom}(U_i Y) \rightrightarrows \prod_{i,j} \mathrm{Hom}(U_i \cap U_j, Y).$$

This follows from our proposition: clearly the image of the first is included in the kernel of the second; to show exactness, we say that it is enough to check continuousness on the U_i's, and use our proposition.

Next, we have

$$\begin{array}{ccccc} \mathrm{Hom}(X,Y) & \to & \prod_i \mathrm{Hom}(U_i Y) & \rightrightarrows & \prod_{i,j} \mathrm{Hom}(U_i \cap U_j, Y) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}(S,R) & \to & \prod_i \mathrm{Hom}(S, \Gamma(U_i, O_{U_i})) & \rightrightarrows & \prod_{i,j} \mathrm{Hom}(B, \Gamma(U_i \cap U_j, O_X)) \end{array}$$

The bottom sequence is exact (check this!). Then the first map (that is, $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(S,R)$ is injective. The middle map is an isomorphism because the U_is are affine. The third map is also injective, and this proves that the first is an isomorphism by a nifty diagram chase. This proves the lemma.

2.2 General case

Now we will try to do the general case.

Let's write $|Z| = |X| \times |Y|$. For any $U \subset X, V \subset Y$ affine open we can write $|U \times V| \hookrightarrow |Z|$; if the U_i cover X and the V_i cover Y then the $U_i \times V_j$ cover Z.

Claim: To prove that $X \times Y$ exists, it is enough to define a topology and sheaf of functions O_Z on $|Z|$ such that $\forall U \subset X, V \subset Y$ affine open sets, then $|U \times V$ is open and $(|U \times V|, O_Z|_{|U \times V|}) \cong U \times V$. This would make $(|Z|, O_Z)$ a prevariety; then to prove it is the product, we just have to construct the map $W \to Z$ locally.

We'll get there next time.