November 6, 2003

1 Obligatory "section" break

Theroem (Chow's Lemma) If X is a complete variety, then there exists a birational surjective map $\pi : Y \to X$ with Y projective.

Pf. We would like to embed X in projective space but we can't. Write $X = \cup U_i$ and choose $U_i \hookrightarrow \mathbb{P}^n$ with the same n for every i, and let $\overline{U_i} \subset \mathbb{P}^n$ be the closure of U_i. Let $U^* = \cap_i U_i$.

$$U^* \hookrightarrow \overline{U_1} \times \ldots \times \overline{U_r} \hookrightarrow \mathbb{P}^n \times \ldots \times \mathbb{P}^n \hookrightarrow \mathbb{P}^N,$$

for large enough N. We know U^* is locally closed in $\overline{U_1} \times \ldots \times \overline{U_r}$ because the diagonal map is closed and the inclusion in the closure is open.

Tak Y to be the closure of U^* in the product of the closures of the U_i. It is clear Y is projective, as it sits in $\mathbb{P}^N$.

Say $\pi : Y \to X$ exists. Then $\Gamma_\pi(Y) \subset X \times Y$ consists of points $(\pi(y), y)$. The graph $\Gamma_\pi(Y)$ determines the morphism π, and we know the graph must be the closure of U^* in $X \times Y$ (U^* is a subset of each).

$$\begin{array}{ccccc} U^* & \hookrightarrow & U^* \times \ldots \times U^* & \hookrightarrow & X \times \overline{U_1} \times \ldots \times \overline{U_m} \\ = & & \downarrow & & \downarrow r_i \\ U^* & \overset{\Delta}{\hookrightarrow} U^* \times U^* \subset X \times \overline{U_i} & & & \end{array}$$

Let $\tilde{Y}$ be the closure of U^* under the two inclusions in $X \times \overline{U_1} \times \ldots \times \overline{U_m}$. Then $r_i(\tilde{Y}) \cap (X \times U_i) = r_i(\tilde{Y} \times (U_i \times \overline{U_i}) = \{(x,x) | x \in U_i\}$

Let $\tilde{Y_i} = \tilde{Y} \cap (X \times \overline{U_1} \times \ldots \times U_i \times \ldots \times \overline{U_m})$. Then $Y_i = Y \cap \overline{U_1} \times \ldots \times U_i \times \ldots \times \overline{U_m}$. Now we know that $\tilde{Y_i} \xrightarrow{p_2} Y_i$. This should be an isomorphism; define an inverse $\sigma_i : Y_i \to \tilde{Y_i}$ defined by $\sigma_i(u_1, \ldots, u_m) = (u_i, u_1, \ldots, u_m)$. The point is that when we have $x, u_1, \ldots, u_m$ then we know $x = u_i$.

Lemma. Let S and T be varieties, such that $V \hookrightarrow S$ and $V \hookrightarrow T$. Consider the map λ which is the composition along $V \overset{\Delta}{\hookrightarrow} V \times V \hookrightarrow S \times T$. Let $\overline{V}$ be the closure of $\lambda(V)$. Then $\overline{V} \cap (S \times V) = \overline{V} \cap (V \times T) = \Delta(V)$.

Pf. It is enough to see that $\Delta(V)$ is closed in both $V \times T$ and $S \times V$. This is because $\Delta(V)$ is a graph of both inclusions $V \hookrightarrow S$ and $V \hookrightarrow T$, and these are morphisms of varieties.

2 Analytic spaces

Let $X \subset \mathbb{P}^n_{\mathbb{C}}$. What is the relationship between the variety X and points of X with the complex topology?

Def. (Grauert-Remmert). An analytic space is a pair $(\mathcal{C}, \mathcal{O}_\mathcal{C})$, where $\mathcal{C}$ is a topological space, and $\mathcal{O}_\mathcal{C}$ is a sheaf of complex-valued functions, such that there is a covering $\mathcal{C} = \cup_i \mathcal{U}_i$ such that $(\mathcal{U}_i \mathcal{O}_\mathcal{C}|_{\mathcal{U}_i}) = D_n = \{(z_1, \ldots, z_n) || z_1| < 1\}$ of $f_1, \ldots, f_n$ analytic functions on D_n. $\mathcal{O}_\mathcal{C}(\mathcal{U}) = \{f : \mathcal{U} \to \mathbb{C} | \exists$ cover $\mathcal{U} = \cup \mathcal{U}_j$ and open $\tilde{\mathcal{U}_j} \subset D_n$ of $\mathcal{U}_j$ such that $f|_{\mathcal{U}_j}$ is the restriction of analytic functions on $\tilde{\mathcal{U}_j}\}$.

$(\mathcal{C}, \mathcal{O}_\mathcal{C})$ is a complex manifold if it is locally isomorphic to (D_n, O_{D_n}).

A morphism $(\mathcal{C}, \mathcal{O}_\mathcal{C}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is a continuous map $g : \mathcal{C} \to \mathcal{Y}$ such that for all open $\mathcal{U} \subset \mathcal{Y}, h \in \mathcal{O}_\mathcal{Y}(\mathcal{U})$, the composite

$$g^{-1}(\mathcal{U}) \xrightarrow{g} \mathcal{U} \xrightarrow{h} \mathbb{C}$$

is in $\mathcal{O}_\mathcal{C}(g^{-1}(\mathcal{U}))$.

Claim. To any prevariety X there is an associated analytic space X_{an} and a continuous map $\pi_X : X_{an} \to X$ such that $\forall U \subset X$ open and $f \in O_X(U)$ the composite

$$\pi_X^{-1}(U) \xrightarrow{\pi_X} U \xrightarrow{f} \mathbb{C}$$

is in $O_{X_{an}}(\pi_X^{-1}(U))$.

Pf. of claim: We can construct $X_{an} \to X$ locally on X.

Let $X \subset \mathbb{A}^n$, $X_{an} = X$ with topology induced by the usual topology on $\mathbb{C}^n$. Also, $O_{X_{an}}$ is the sheaf of holomorphic functions. So that is what we have locally, and to get it globally, we glue things. This gets too complicated for this quick overview.

Let X be the affine line with double origin. What is X_{an}? Basically, we get $\mathbb{C}$ with the origin doubled.

Theorem Let X be a prevariety. Then

1. X is a variety $\iff$ X_{an} is Hausdorff.
2. X is a complete variety $\iff$ X_{an} is compact (which includes Hausdorff).

We prove this via the following Lemma, which we actually won't prove now.

Lemma. Let X be a prevariety, $U \subset X$ an open dense set. Then $U_{an} \subset X_{an}$ is dense.

Using this, we prove the theorem. (1): $\overline{\Delta(X)}$ is dense in either topology. If $\Delta(X) \neq \overline{\Delta(X)}$ then $\delta(X_{an})$ has a limit point: it converges to x, y with $x \neq y$ which gives two points that fail Hausdorff.

(2): If X is projective we're done (we just need that $\mathbb{P}^n_{an}$ is compact, which it is, because it's just $\mathbb{CP}^n$.) In general, we have $Y \xrightarrow{\pi} X$ where Y is projective and similarly $Y_{an} \xrightarrow{\pi} X_{an}$ but now Y_{an} is compact, and X_{an} is Hausdorff and the map is surjective! Thus X_{an} is compact.

Question: Given $(\mathcal{C}, \mathcal{O}_\mathcal{C})$ when is it an X_{an}. This is true if $\mathcal{C} \hookrightarrow \mathbb{P}^n_{an}$ (Serre). Example: let Λ be a lattice $\mathbb{Z} \oplus \mathbb{Z}w$, then $\mathbb{C}/\Lambda$ is algebraic.

Thm. (Riemann): Every compact Riemann surface is algebraic. There is actually an equivalence of categories between compact Riemann surfaces and complete varieties X of dimension 1 where for all $x \in X$ the ring $O_{X,x}$ is regular.