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Thm. There is an equivalence of categories between complete smooth curves (with nonconstant morphisms) and finitely generated field extensions $k \to K$ of transcendence degree 1, defined by $C \mapsto k(C)$.

Main point: for any point in C, $O_{C,p} \subset k(C)$ is a DVR. If $p, q \in C$ then $O_{C,q} = O_{C,p}$ implies $p = q$, so the map sending $|C|$ to the set of DVRs in $k(C)$ is a bijection.

So given $k \to K$, consider (C_K, O_{C_K}) where C_K is the set of DVRs in K. Topology is given so that open sets are the complements of finite sets.

Prop. (C_K, O_{C_K}) is a complete smooth curve. 1. Every $k \to K$ is isomorphic to $k \to k(C)$ for some curve C. 2. If C_1, C_2 are smooth, complete curves, then $\mathrm{Hom}(C_1, C_2) \cong \mathrm{Hom}(k(C_2), k(C_1))$.

(1) we already proved. (2) is called "full faithfulness." We prove it as follows.

Suppose we have $f, g : C_1 \to C_2$ such that the induced maps $k(C_2) \to k(C_1)$ are the same, then $f = g$. We know if f and g agree on a dense open set in a variety then they are equal, and they must for the maps on the function fields to be the same.

Suppose we have a map $\rho : k(C_2) \to k(C_1)$. If we have $p \in C_1$, there is a valuation ring $O_{C,p} \subset k(C_1)$, so we take the inverse image of this under ρ and this will also be a DVR (it's an intersection with a field), so we take the point it corresponds to. Thus, we get our map ρ^*. This is a morphism.

Theorem. Every complete, smooth curve is projective.

Pf. Chow's lemma tells us we can get a map $\pi : C' \to C$ which is birational where C' is projective, of dimension 1. Choose $p \in C'$. Then we have $O_{C,\pi(p)} \subset O_{C',p} \subset k(C)$. So $O_{C',p}$ dominates $O_{C,\pi(p)}$. Valuation rings are exactly those that are maximal with respect to domination. Thus, $O_{C',p} = O_{C,\pi(p)}$ so $O_{C',p}$ is regular so C' is smooth. Thus, π is an isomorphism, so C is in fact projective.

Remark. When $k = \mathbb{C}$, TFAE categories:

1. (Compact Reimann surfaces)
2. (Complete smooth curves)
3. (Finitely generated field extensions $\mathbb{C} \to K$ of transcendence degree 1)

We have shown $2 \cong 3$, and we have a functor from 2 to 1 defined by $C \mapsto C_{an}$, and we can map S to its field of meromorphisms.

1 Now what?

We may want to classify complete smooth curves over $\mathbb{C}$.

Def. Let C be a complete, smooth curve. The *genus* of C is the number $g(C) = \dim_k H^0(C, \Omega_C)$. This is the "number of holes" in a compact Reimann surface.

Ex. $g(\mathbb{P}^1) = 0$. $\mathbb{P}^1$'s coordinates are $[x : y]$. Now $U_{y\neq 0} = k[u]$ where $u = (x/y)$. We also have $U_{x\neq 0} = k[v]$ where $v = (y/x)$. Then $\Omega^1_{\mathbb{P}^1}(U_{y\neq 0}) = k[u] \cdot du$, and similarly $\Omega^1_{\mathbb{P}^1}(U_{x\neq 0}) = k[v] \cdot dv$.

$\Omega^1_{\mathbb{P}^1}(U_{yx\neq 0}) = k[u^{\pm}] \cdot du$.

Now $k[u^{\pm}] \cdot du \cong k[v^{\pm}] \cdot dv$. Clearly $u \mapsto 1/v$ and vice versa. Similarly, $du \mapsto -\frac{1}{v^2}dv$.

Say $w \in H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1})$. Say $w|_{U_{y\neq 0}} = f(u)du$ and $w|_{U_{x\neq 0}} = g(v)dv$, then we get $f(1/v)(-1/v^2)dv$, which can't be a polynomial in v, so $w = 0$ and thus the dimension is 0.

Ex. Let $E = V(y^2z = x^3 + z^3) \subset \mathbb{P}^2$. We know this is an elliptic curve and should be just like taking $\mathbb{C}/\Lambda$ for some lattice Λ.

Consider $E_{zy\neq 0} = D(y)$ .. good lord, I have no idea what he's doing.

Blah blah blah, lots of totally disorganized bull.

$g(E) \geq 1$. It is in fact 1, but he doesn't prove this, despite that he did "prove" it had genus at least 1.

2 How to study curves

1. Embed $C \hookrightarrow \mathbb{P}^n$.
2. Map $C \mapsto \mathbb{P}^1$. (Corresponds to choosing an embedding $k(t) \to k(C)$)

We're really talking about $h_{\mathbb{P}^n}$: in the first, we consider $h_{\mathbb{P}^n}(C)$ and in the second, we consider $h_{\mathbb{P}^1}(C)$.

Let's think about how to define maps $C \to \mathbb{P}^n$ $\mathbb{P}^n$ is the set of lines in k^{n+1}. That is, C will be a *continuous family of lines.*

Def. A *line bundle* or *invertible sheaf* or *line sheaf* is a sheaf of O_X-modules such that $\exists X = \cup U_i$ such that $\mathcal{Y}|_{U_i} \cong O_{U_i}$.

Ex. Say C is a smooth curve. Then Ω^1_C is a line bundle. Choose a point $p \in C$, then $\Omega^1_{C,p} \otimes_{O_{C,p}} k = \Omega^1_C(p)$ which has dimension 1. Choose $w \in \Omega^1_{C,p}$ mapping to a basis for $\Omega^1_C(p)$. Say U affine, $p \in U$, and $R = \Gamma(U, O_U)$.

Consider the map $R_m \twoheadrightarrow \Omega^1_{C,p} = \Omega \otimes R_m$. THere is an $f \in R$ such that $f(p) = 0$ and $R_f \twoheadrightarrow \Omega \otimes R_f$. Claim: this is an isomorphism.