Lecture 1: Course Introduction, Zariski topology

Some teasers So what is algebraic geometry? In short, geometry of sets given by algebraic equations. Some examples of questions along this line:

1. In 1874, H. Schubert in his book *Calculus of enumerative geometry* proposed the question that given 4 generic lines in the 3-space, how many lines can intersect all 4 of them.

   The answer is 2. The proof is as follows. Move the lines to a configuration of the form of two pairs, each consists of two intersecting lines. Then there are two lines, one of them passing the two intersection points, the other being the intersection of the two planes defined by each pair. Now we need to show somehow that the answer stays the same if we are truly in a generic position. This is answered by *intersection theory*, a big topic in AG.

2. We can generalize this statement. Consider 4 generic polynomials over $\mathbb{C}$ in 3 variables of degrees d_1, d_2, d_3, d_4, how many lines intersect the zero sets of each polynomial? The answer is $2d_1d_2d_3d_4$. This is given in general by "Schubert calculus."

3. Take $\mathbb{C}^4$, and 2 generic quadratic polynomials of degree two, how many lines are on the common zero set? The answer is 16.

4. For a generic cubic polynomial in 3 variables, how many lines are on the zero set? There are exactly 27 of them. (This is related to the exceptional Lie group E_6.)

Another major development of AG in the 20th century was on counting the numbers of solutions for polynomial equations over $\mathbb{F}_q$ where $q = p^n$. Here's an example question: $y^2 = x^3 + 1$. The answer, assuming $p = 2 \pmod 3$ and $p \neq 2$, is p^n if n odd, and $(p^{n/2} - (-1)^{n/2})^2 - 1$ otherwise.

A third idea is to study "the shape" (i.e. the topology) of the set of solutions of a system of polynomial equations. For instance, if we consider $y^2 = x^3 = ax + b$ in $\mathbb{C}^2$, this will yield ' $T^1 \times T^1$ with a point removed. Another example: if we have a generic degree 4 equations in $\mathbb{C}^3$ (a K3 surface), then the rank of H^2 (second cohomology) of this space is 22.

Algebraic Varieties We always assume working over some algebraically closed field k. Algebraic varieties are glued from affine varieties.

For instance, consider $\mathbb{A}^n_k = k^n$. It comes with the coordinate ring $R = k[x_1, \ldots, x_n] = k[\mathbb{A}^n]$, which is a commutative k-algebra. How do we recover k^n from $R = k[x_1, \ldots, x_n]$? The first answer, the tautological one, is that $k^n \cong \mathrm{Hom}_{\mathrm{k-alg}}(R, k)$. Namely, given a point $(a_1, \ldots, a_n)$, we can map x_i to a_i. However, there is a second answer: that k^n is the set of maximal ideals of R, which we denotes as Spec R.

To see this, first note that k^n embeds into Spec R. This is simple: you just map each point $(a_1, \ldots, a_n)$ to the kernel of the map $R \to \mathbb{C}$ given by $x_i \mapsto a_i$. Surjectivity is less trivial: it is the essential Nullstellensatz.

Theorem 1.1 (Essential Nullstellensatz)**.** *If K/k is a field extension, and K is a finitely generated k-algebra, then K/k is algebraic. In particular, if $k = \overline{k}$, then $K = k$.*

Assuming this statement, and that $\mathfrak{m}$ is an maximal ideal, then $K = R/\mathfrak{m}$ is a field, and it contains k, so $K = k$, thus $R = k \oplus \mathfrak{m}$, thus for each x_i there's some a_i such that $x_i - a_i \in \mathfrak{m}$, so $\mathfrak{m}$ is the kernel of $x_i \mapsto a_i$.

Proof of essential Nullstellensatz. Let's prove this when k is not countable. (Note in particular this excludes the case of $\overline{\mathbb{Q}}/\mathbb{Q}$.) Assume $t \in K$ is not algebtraic over k, then $k(t) \subseteq K$. Note that $(t-a)^{-1} \in k(t)$ for each $a \in k$. But K is at most countably dimensional as a vector space over k, so $(t - a_i)^{-1}$ are linearly dependent, so there is some relation $\sum_i b_i(t - a_i)^{-1} = 0$. Then after getting rid of the denominator by multiplying by $\prod(t - a_i)$, we obtain a polynomial having t as a zero. □

Definition 1. *A Zariski closed subset in k^n is a set given by the zero set of polynomial equations.*

Theorem 1.2. *Zariski closed subsets in k^n are in bijection with radical ideals in $R = k[x_1, \ldots, x_n]$. (Recall that I is a radical ideal if R/I has no nilpotents.)*

Proof. An ideal I maps to Z_I, the set of common zeroes of elements of I. A Zariski closed set Z goes to $I_Z = \{f \mid f|_Z = 0\}$. Clearly $Z_{I_Z} = Z$. Need to check $I_{Z_I} = I$. Let's first consider $Z_I = \emptyset$, then we want $I = R$. If $I \neq R$, then choose $\mathfrak{m} \supseteq I$, then we know that $\mathfrak{m}$ corresponds to some point $a \in Z_I$, contradiction. Now in general, if $f|_{Z_I} = 0$, then $f^n \in I$ for some n. Consider the localization $R_{(f)} = R[t]/(1-ft)$, which can also be written as $\{p/f^n \mid p \in R\}$ mod out a certain equivalence relation $\sim$. Clearly there is an embedding $R \to R_{(f)}$, and hence $\mathrm{Spec}(R_{(f)}) \hookrightarrow \mathrm{Spec}(R)$, where the first is the set of $\{\mathfrak{m} \in R \mid f \notin \mathfrak{m}\}$, thus $IR_{(f)}$ is not contained in a maximal ideal, i.e. $IR_{(f)} = R_{(f)} \implies p/f^n$ for some $p \in I$, then $f^n = p \in I$. $\square$

Corollary 1. *There is a Zariski topology on $\mathbb{A}^n$, where the closed sets are the Zariski closed sets.*

Proof. One just need to check the condition for union and intersection. $\square$

Let's introduce some notions to begin with. (This can be found as [Kem93], Section 1.1 and 1.2.) A *space with function* is a topological space X, where we attach to each open set U a k-albegra, denoted by $k[U]$ and called the regular functions on U. They need to satisfy some conditions:

1. If $U = \bigcup_\alpha U_\alpha$, and f is regular on U, then $f|_{U_\alpha}$ is regular on U_α for each α.
2. If f is regular on U, then $D(f) = \{x \in U \mid f(x) \neq 0\}$ is open and $1/f$ is regular on $D(f)$.

A morphism between spaces with functions is a map $f : X \to Y$ between spaces, such that if g is regular on U, then f^*g is regular on $f^{-1}(U)$. The map $f \mapsto f^*$ gives us a mapping $* : \mathrm{Morphism}(X, Y) \to k - \mathrm{Hom}(k[Y], k[X])$.

Definition 2. *An affine variety is a space with functions Y such that $*$ is bijective for every X and $k[Y]$ a finitely generated k-algebra.*

References

[Kem93] George Kempf. *Algebraic varieties*. Vol. 172. Cambridge University Press, 1993.

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