Lecture 2: Affine Varieties

Side Remark Recall that we introduced three types of questions in the last lecture: counting over $\mathbb{C}$, counting over $\mathbb{F}_q$ and the slope of the set of solutions over $\mathbb{C}$. It is worth pointing out that there is indeed a connection between the two latter types, as sketched out by the Weil conjectures.

Last time we defined Spec A, where A is a finitely generated k-algebra with no nilpotents. Namely, Spec $A = \mathrm{Hom}_{k-alg}(A, k) = \{\text{maximal ideals in } A\}$. Zariski closed set are defined in [Kem93]. Now recall that there is a bijection between Zariski closed subsets of Spec A and the radical ideals of A. Suppose Z_1, Z_2 correspond to I_1, I_2, then $Z_1 \cup Z_2$ corresponds to $I_1 \cap I_2$. Note that $I_1 + I_2$ may not be reduced even if Z_1, Z_2 are varieties. For instance, let $A = k[x, y]$, $I_1 = (y - x^2)$, $I_2 = (y)$, then $A/(I_1 + I_2) = k[x]/x^2$.

Theorem 1.1. *Let $k[U]$ denote functions associated with the set U, as specified in last lecture. Then $k[\text{Spec } A] \cong A$.*

Proof. (This was done in [Kem93], Section 1.3-1.5.) Recall that as a set, Spec A is $k-\mathrm{Hom}(A, k)$, because each maximal ideal is the kernel of a homomorphism $A \to k$ and vice versa. So there's a map $\phi : A \to k[\text{Spec } A]$ given by $a \mapsto (x \mapsto x(a))$, which we shall prove to be a bijection.

We first want the topological structure on Spec A. This is given by $Z(I) = \{x \in \text{Spec } A \mid i(x) = 0\ \forall i \in I\}$, where I is a subset of A. One can directly check that this gives a topology on Spec A. Next we need to make it a space with functions. The construction is given as: $k[U] = \{f : U \to k \mid \exists (U_\alpha, a_\alpha, b_\alpha), \bigcup_\alpha U_\alpha = U, f|_{U_\alpha} = \phi(a_\alpha)/\phi(b_\alpha), \phi(b_\alpha)(x) \neq 0\ \forall x \in U_\alpha\}$.

To show injectivity, let $a \neq 0 \in A$, then we need to find some $x : A \to k \in \text{Spec } A$ such that $\phi(a)(x) = x(a) \neq 0$. To do so we'd need the following fact, the proof of which is standard commutative algebra:

Lemma 1 (Noether Normalization)**.** *Given A a finitely-generated k-algebra, there exists some algebraically independent elements $X_1, \ldots, X_d$ over k such that A is a finitely generated $k[X_1, \ldots, X_d]$-module.*

Apply this fact with the localization $A_{(a)}$, which is nonempty because A has no nilpotent (otherwise if $1 = 0$ in the localization ring, then $a^n = a^n \cdot 1 = 0$), and is finitely generated as we just need to add $1/a$ to A. Thus we get some $X_1, \ldots, X_d$ such that $A_{(a)} \supseteq B = k[X_1, \ldots, X_d]$, then there is a surjection $\psi : k - \mathrm{Hom}(A_{(a)}, k) \to k - \mathrm{Hom}(B, k)$. Let $\varphi \neq 0 \in k - \mathrm{Hom}(B, k)$, and let $\psi(\tilde{\varphi}) = \varphi$, and let $x = A \hookrightarrow A_{(a)} \xrightarrow{\tilde{\varphi}} k$, then $1 = x(1) = \tilde{\varphi}(a)\tilde{\varphi}(1/a) = x(a)\tilde{\varphi}(1/a)$, so $x(a) \neq 0$.

Now we need surjectivity. Take $f \in k[\text{Spec } A]$ and we need to show it is in A. Assume the data is given by $(U_\alpha, a_\alpha, b_\alpha)$, where we can assume that each $U_\alpha = D(c_\alpha)$. By the replacement $a_\alpha \mapsto a_\alpha c_\alpha, b_\alpha \mapsto b_\alpha c_\alpha$, one can assume that $U_\alpha = D(b_\alpha)$. Since the $D(b_\alpha)$ sets cover Spec A, we know that the ideal generated by $\{b_\alpha^2\}_\alpha$ corresponds to empty set, thus by Nullstellensatz (c.f. [Kem93], Theorem 1.4.5), there must be some finite set $b_1, \ldots, b_m$ and some constants $z_1, \ldots, z_m \in A$ such that $\sum_{i=1}^m z_i b_i^2 = 1 \in A$. Now $b_\alpha^2 f$ agrees with $a_\alpha b_\alpha$ both on U_α and its complement, so they are equal in A, which means $f = f \cdot 1 = \sum_i z_i (f b_i^2) = \sum_i z_i a_i b_i \in A$. □

Note this last part can also give us the following:

Proposition 1. Spec A *is quasi-compact for any commutative ring A.*

Proof. Take a covering $X = \bigcup U_\alpha$, then can pick $U_{f_\alpha} \subseteq U_\alpha$, then we have $(f_\alpha) = 1$, and thus there's a finite subset $(f_{d_1}, \ldots, f_{d_n}) = 1$. □

What we really want to say is:

Theorem 1.2. *Given a space of functions X, X is an affine variety if and only if $X =$ Spec A for a finitely generated commutative ring A with no nilpotents.*

Proof. Let's show that Spec A is affine; the other direction will be done in the next lecture. Let X be any space with functions, then we need to show that $* : \mathrm{Morphism}(X, \text{Spec } A) \to k - \mathrm{Hom}(A, k[X])$ is injective and surjective. For injectivity, let $f : X \to \text{Spec } A$ be a morphism and let x be any point on X, then $\delta_{f(x)}$, the evaluation map at $f(x)$, is given by $\delta_{f(x)}(a) = a(f(x)) = (f^*a)(x)$ for $a \in A$, i.e. $f(x)$, equivalently $\delta_{f(x)}$, is specified by x and f^*. On the other hand, define $*^{-1}$ by $\delta_{*^{-1}(g)(x)} = \delta_x \circ g$, then one can check this gives a well-defined inverse to $*$ and thus $*$ is bijective. $\square$

Definition 1. *An algebraic variety over k is a space with functions which is a finite union of open subspaces, each one is an affine variety.*

Lemma 2. *A closed subspace in an affine variety is also affine, and global regular functions restrict surjectively.*

Proof. $X = \text{Spec } A$, $Z = Z_I$, I is a radical. Then $Z_I \cong \text{Spec}(A/I)$. Surjectivitly follows from the fact that $k[\text{Spec } A] = A$. $\square$

Corollary 1. *A closed subspace of a variety is a variety.*

Theorem 1.3 (Hilbert Basis Theorem)**.** *$k[x_1, \ldots, x_n]$, and hence any finitely generated k-algebra is Noetherian.*

Corollary 2. *An algebraic variety is a Noetherian topological space (that is, every descending chains of closed subsets terminate; equivalently, every open subset is quasicompact).*

Corollary 3. *An open subspace of an algebraic variety is an algebraic variety. (Contrast with affine variety.)*

Proof. Need to check that an open subset of an affine variety is covered by finitely many affine varieties. This follow from quasi-compactness. $\square$

Combine the two corollaries above, we see that a locally closed subspace (intersection of open and closed) of an algebraic variety is again a variety. However, the union of an open set and a closed set need not be a variety. For an counterexample, consider $(\mathbb{A}^2 - \{x = 0\}) \cup \{0\}$.

Definition 2 (Projective Space)**.** *Topologically, the projective space $\mathbb{P}^n$ is given by the quotient topology $\mathbb{A}^{n+1} - \{0\}/(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n) \forall \lambda \neq 0$. A function on $U \subseteq \mathbb{P}^n$ is regular if its pullback by $\mathbb{A}^{n+1} - \{0\} \xrightarrow{\pi} \mathbb{P}^n$ is regular on $\pi^{-1}(U)$.*

References

[Kem93] George Kempf. *Algebraic varieties*. Vol. 172. Cambridge University Press, 1993.