Lecture 3: Projective Varieties, Noether Normalization

Review of last lecture Recall that Spec $A = \mathrm{Hom}_{k-\mathrm{alg}}(A, k)$. Let I and J be ideals of A. The following question was asked while we were discussing the topology on Spec A.

Question 1. *When do we have that $IJ = I \cap J$?*

Answer (From MO.) When $\mathrm{Tor}_1^A(A/I, A/J) = 0$ (Tor_1^A is the derived functor of tensor products $\otimes_A$). For example, we can take $A = k[V]$, $I = Z_W$, and $J = Z_U$, where U and W are subspaces of a vector space V such that $U + W = V$.

Last time, we started the proof of the following theorem:

Theorem 1.1. *Let X be a space with functions. Then, X is affine if and only if $X =$ Spec A for some finitely generated k-algebra A with no nilpotents.*

Proof. The proof that X is affine if $X =$ Spec A for some A was done in the last lecture. It remains to check that $X =$ Spec A for some A if X is affine. Assume that X is affine. Note that $k[X] =: A$ is a finitely generated k-algebra which is a nilpotent ring (since it is an algebra of functions). Take $X' =$ Spec A. Since X is affine, the isomorphism $k[X] = A \cong k[X']$ gives a map $X' \longrightarrow X$. We also know that X' is affine. So, we get a map $X \longrightarrow X'$. Applying the affineness of X and X' to the two compositions, we see that these are inverse isomorphisms and $X =$ Spec A.

□

Closed subvarieties of $\mathbb{P}^n$ At the end of last lecture, we defined the projective space $\mathbb{P}^n_k$ over a field k and described the regular functions on it. Recall that $\mathbb{P}^n_k = \mathbb{A}^{n+1} \setminus \{0\}/k^\times$. This space has an affine cover $\mathbb{P}^n_k = \bigcup_{i=0}^n \mathbb{A}^n_i$, where $\mathbb{A}^n_i = \{(x_0, x_1, \ldots, x_n) : x_i \neq 0\}/k^\times \cong \{(x_0, x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)\}$. Note that it is a disjoint union of locally closed subsets since $\mathbb{P}^n_k \setminus \mathbb{A}^n_k \cong \mathbb{P}^{n-1}_k$ and $\mathbb{P}^n = \coprod_{i=0}^n S_i$, where S_i is locally closed and isomorphic to $\mathbb{A}^i$.

Example 1. If $k = \mathbb{C}$, we can take $\mathbb{P}^n_\mathbb{C}$ to be a topological space with the complex (classical) topology. Since it a union of cells of even real dimension, we have

$$\dim H^i(\mathbb{P}^n_\mathbb{C}) = \begin{cases} 1 & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

Now consider the antipodal map $S^{2n+1} \twoheadrightarrow \mathbb{P}^n_\mathbb{C}$. Since this map is continuous and onto, it follows that $\mathbb{P}^n_\mathbb{C}$ is compact.

Example 2. Suppose that $k = \mathbb{F}_q$. Then, we have $|\mathbb{P}^n_k| = \sum_{i=0}^n q^i = \dfrac{q^{n+1} - 1}{q - 1} := [n]_q$ (q-analogues).

Definition 1. *An algebraic variety is* projective *if it is isomorphic to a closed subvariety of a projective space.*

Remark 1. If X is a projective variety over $\mathbb{C}$, then X taken in the classical topology is compact.

Definition 2. *An algebraic variety is* quasiprojective *if it is a* locally closed *subvariety in a projective space.*

Most of the things we use have this property.

Remark 2. It is important to check whether we are working with the Zariski topology or the classical topology. If a set is closed in the Zariski topology, it is also closed in the classical topology over $\mathbb{C}$ since polynomials are continuous functions. However, a set which is closed in the classical topology may not be Zariski closed.

Next, we describe the closed subvarieties of $\mathbb{P}^n$. Note that closed subvarieties in $\mathbb{P}^n$ correspond to the $k^\times$-invariant subvarieties of $\mathbb{A}^{n+1} \setminus \{0\}$. Let $V = k[x_0, \ldots, x_n]$ and $X \subset \mathbb{P}^n$ be a closed subvariety. Then, V is a graded vector space $V = \bigoplus_n V_n$, where V_n is the set of homogenous polynomials of degree n. Now consider the action of $t \in k^\times$ on V. Since we have $t|_{V_n} = t^n \mathrm{Id}$, we have that $f \in V$ vanishes on X if and only if all of its homogeneous components f_n vanish on X. Thus, we have that I_X is a homogeneous (= graded) ideal. If k is algebraically closed, we have the following correspondence ([SH77, p. 41-42]):

$$\text{closed subvarieties in } \mathbb{P}^n \longleftrightarrow \text{radical (nonunital) homogeneous (= graded) ideals in } k[x_0, \ldots, x_n]$$

We can also obtain closed subvarieties of $\mathbb{P}^n$ by taking projective closures of closed subvarieties X of $\mathbb{A}^n$. Recall that there is an open $\mathbb{A}^n_0 = \{(x_0, \ldots, x_n) : x_0 \neq 0\} = \mathbb{A}^n \subset \mathbb{P}^n$. For closed $X \subset \mathbb{A}^n$, we get $\overline{X}$, which is the closure of X in $\mathbb{P}^n$. If $P \in k[Y_1, \ldots, Y_n]$ vanishes on X, then $\tilde{P} = x_0^d P\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\right)$ vanishes on $\overline{X}$, where $d = \deg P$. Note that $P = \tilde{P}(1, Y_1, \ldots, Y_n)$. For example, if $P = X^3 - Y^2 - Y + 1$, then $\tilde{P} = X^3 - ZY^2 - Z^2Y + Z^3$. We also have that $I_{\overline{X}} = (\tilde{P} : P \in I_X)$.

Example 3 (Linear subvarieties in $\mathbb{P}^n$)**.** *If I_X can be generated by linear polynomials, then X can be sent to $\{(x_0 : \cdots : x_n) : x_{i+1} = \cdots = x_n = 0\}$ by a linear change of variables (i.e. invariant matrices acting on $\mathbb{P}^n$). Let $X \subset \mathbb{P}^2$ be a degree d irreducible curve and $I_X = (P)$, where $P \in k[X, Y, Z]$ is a degree d irreducible polynomial.*

Case 1: $d = 1$ *This is the case where $X = \mathbb{P}^1$.*

Case 2: $d = 2$ *(char $k \neq 2$) Claim: $X \cong \mathbb{P}^1$ again. Proof sketch: By linear algebra, all irreducible degree 2 polynomials in 3 variables are permuted transitively by a linear change of variables. Without loss of generality, we can assume that $P = XY - Z^2$. On $\mathbb{A}^2$ ($Z \neq 0$), we get $(XY = 1) \cong \mathbb{A}^1 \setminus \{0\}$. Exercise: Finish this.*

Here is another construction of the isomorphism $X \cong \mathbb{P}^1$. Fix $x \in X$. Consider the following correspondences:

$$\{\textit{lines in } \mathbb{P}^1 \textit{ passing through } x\} \leftrightarrow \{\textit{dim. 2 subvarieties of } \mathbb{A}^3 := V \textit{ containing } L_x\} \leftrightarrow \{\textit{dim. 1 subvarieties in } V/L_x\}$$

Note that the last set is isomorphic to $\mathbb{P}^1$. Here, $L_x \subset \mathbb{A}^3$ is the set of lines passing through x. Now construct the map $X \setminus x \longrightarrow \mathbb{P}^1$ sending y to the line passing through x and y. Exercise: Finish this.

Case 3: $d = 3$ *X is* not *necessarily isomorphic to $\mathbb{P}^1$ in this case. For example, suppose that X is an elliptic curve. Claim: By a linear change of variables, we can get X to the Weierstrass normal form $y^2 = x^3 + ax + b$. The closure of this curve in $\mathbb{P}^2$ intersects the line at infinity at 1 point:*

$$ZY^2 = X^3 + aXZ^2 + bZ^3$$
$$Z = 0 \Rightarrow X = 0$$
$$\textit{Intersection point}: (0 : 1 : 0)$$

Note that $\mathbb{P}^1$ also has one point at infinity. Comparing the set regular functions on the affine parts of X and $\mathbb{P}^1$ and noting that $k[X, Y]/(Y^2 - X^3 - aX - b)$ is not *generated by one element (has a filtration with the associated graded ring $k[X, Y]/(Y^2 = X^2)$), we find that $X \not\cong \mathbb{P}^1$.*

Noether normalization lemma and applications

Theorem 1.2. *(Noether normalization lemma)*
Let A be a finitely generated k-algebra, where k is any *field (not necessarily algebraically closed). Then, we can find $B \subset A$ such that $B \cong k[x_1, \ldots, x_n]$ for some n and A is finitely generated as a B-module.*

Remark 3. Here is a "geometric" version of the theorem which has to do with subvarieties in affine space:

If $B \subset A$ and A is a finitely generated B-module, then the map Spec $A \longrightarrow$ Spec B is onto and has finite fibers.

We will prove the theorem in the case where k is infinite.

Lemma 1. *Take $P \in k[x_1, \ldots, x_n]$ be a nonconstant polynomial and let $d = \deg P$. There is a linear change of variables such that P has for form $x_n^d +$ (terms of $\deg_{x_n} < d$).*

Proof. Write $x_i = x_i' + \lambda_i x_n'$ for $1 \leq i \leq n-1$ and $x_n' = \lambda_n x_n$. If $d = \deg P$ and $P = P_d +$(terms of $\deg < d$), then $P(x_i) = x_n^d P_d(\lambda_1, \ldots, \lambda_n) +$ (terms of $\deg_{x_n} < d$). Thus, we would like to find $\lambda_1, \ldots, \lambda_n$ such that $P_d(\lambda_1, \ldots, \lambda_n) = 1$. Since P_d is homogeneous, it suffices to show that there exist $\mu_1, \ldots, \mu_n$ such that $P_d(\mu_1, \ldots, \mu_n) \neq 0$. Thus, the proof reduces to the following claim:

Claim : A nonzero polynomial over an infinite field takes nonzero values.

This can be proved using induction in number of variables. □

Now we begin the proof of the Noether normalization lemma.

Proof. Since A is finitely generated, we have a surjection $\phi : k[x_1, \ldots, x_n] \twoheadrightarrow A$. We use induction on n. Let $I = \ker \phi$. If $I = (0)$, we are done. Now suppose that $I \neq (0)$. Take $0 \neq P \in I$. By the lemma above, we can assume without loss of generality that $P = x_n^d +$ (terms of $\deg_{x_n} < d$). Note that $k[x_1, \ldots, x_n]/(P) \twoheadrightarrow A$ and $k[x_1, \ldots, x_n]/(P)$ is finite over $k[x_1, \ldots, x_{n-1}]$. Let $A' = \phi(k[x_1, \ldots, x_{n-1}])$. Applying the induction assumption to A', there exists $B \cong k[x_1, \ldots, x_m]$ such that A' is finite over B. Since A is finite over A', A is finite over B and we are done. □

Next, we can show that $k[x_1, \ldots, x_n]$ is Noetherian.

Proposition 1. *(Hilbert basis theorem) $k[x_1, \ldots, x_n]$ is Noetherian.*

Proof. It is enough to check that every ideal is finitely generated. As above, we use induction on n. Let I be a nonzero ideal of A and $0 \neq P$ be an element of I. Without loss of generality, we can assume that $A/(P)$ is finite as a module over $k[x_1, \ldots, x_{n-1}]$. Since $k[x_1, \ldots, x_{n-1}]$ is Noetherian by induction, every submodule of $A/(P)$ is finitely generated over $k[x_1, \ldots, x_{n-1}]$. Hence, $I/(P)$ is finitely generated, which implies that I is finitely generated. □

We need another result in order to finish the proof of the "essential Nullstellensatz" from the first lecture.

Lemma 2. *(Nakayama lemma)*
Let M be a finitely generated module over a commutative ring A. If I is an ideal of A such that $IM = M$, then there exists $a \in A$ such that $aM = 0$ and $a \equiv 1 \pmod{I}$.

Proof. Let $\{m_i\}$ be generators of M. Then, $m_i = \sum a_{ij} m_j$, where $a_{ij} \in I$. Then, we can set $a = \det(1 - a_{ij})$. □

Finally, we can finish the proof of the essential Nullstellensatz.

Theorem 1.3. *("essential Nullstellensatz") Let A be a finitely generated k-algebra. If A is a field, then A/k is algebraic.*

Proof. Since A is a finitely generated k-algebra, it follows from the Noether normalization lemma that there exists $B \cong k[x_1, \ldots, x_n]$ such that $A \supset B$ and A is finitely generated as a B-module. If $n = 0$, we are done since A/k would be a finite extension, which must be algebraic. Suppose that $n \geq 1$. Then, $A \supset \mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of B. It follows from Nakayama's lemma that $\mathfrak{m}A \neq A$. Otherwise, there exists $b \in B$ such that $bA = 0$ and $b \equiv 1 \pmod{\mathfrak{m}}$. This would imply that $bB = 0 \Rightarrow B/\mathfrak{m} = 0$, which is impossible since $\mathfrak{m} \subsetneq B$. Since A has a proper ideal $\mathfrak{m}A$, it is not a field. □

Irreducibility Here is a list of some definitions and properties of topological spaces which will be discussed in more detail in the next lecture.

Definition 3. *A topological space is* irreducible *if any two nonempty open subsets intersect. Equivalently, it is not a union of two proper closed subsets. Another equivalent definition is a space where a nonempty open subset is dense (sort of opposite to Hausdorff...).*

Remark 4. An irreducible topological space is connected, but a connected space is not necessarily irreducible.

Remark 5. Every variety is a union of irreducible pieces.

Proposition 2. *Spec A is irreducible if and only if A has no zerodivisors.*

Definition 4. *A* component *of a topological space is a maximal irreducible closed subset.*

Proposition 3. *A Noetherian topological space is the union of its components (finite in number).*

Corollary 1. We have the following correspondences:

$$\text{Irreducible closed subsets in Spec } A \leftrightarrow \text{Prime ideals in } A$$

$$\text{Components } \leftrightarrow \text{minimal prime ideals (i.e. prime ideals not containing any other prime ideals)}$$

Corollary 2. $0 = \bigcap(\text{minimal prime ideals})$.

References

[SH77] Igor Rostislavovich Shafarevich and Kurt Augustus Hirsch. *Basic algebraic geometry*. Vol. 1. Springer, 1977.

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