Lecture 5: More on Finite Morphisms and Irreducible Varieties

Lemma 1. *Let $f : X \to Y$ be a finite map of varieties and $Z_1 \subsetneq Z_2$ irreducible subvarieties of X. Then $f(Z_1) \subsetneq f(Z_2)$.*

Proof. We can assume WLOG that $f : X = Spec(A) \to Spec(B) = Y$ is surjective and $Z_2 = X$. Pick a nonzero function $g \in I(Z_1)$. Since f is finite, the ring map $B \to A$ turns A into a finitely-generated B-module. In particular, the B-subalgebra of A generated by g is finitely-generated as a B-module. Hence, $g^n = \sum_{i=0}^{n-1} h_i g^i$ for some natural number n and $h_0 \neq 0$. Since $h_0 = g^n - \sum_{i=1}^{n-1} h_i g^i$ vanishes on Z_1, h_0 vanishes on $f(Z_1)$. □

Lemma 2. *If $f : X \to Y$ is a finite surjection of varieties, then $dim(X) = dim(Y)$.*

Proof. Let $X_0 \subsetneq X_1 \subsetneq ... \subsetneq X_n$ be any chain of non-empty irreducible closed subsets of X. Set $Y_i = f(X_i)$. Since f is continuous, $\{Y_i\}$ are irreducible and since f is finite $\{Y_i\}$ are closed. By the previous lemma, the sequence $Y_0 \subset ... \subset Y_n$ is strictly increasing. Hence, $dim(Y) \geq dim(X)$. Conversely, let $Y_0 \subsetneq Y_1 \subsetneq ... \subsetneq Y_m$ be a chain of non-empty irreducible closed subsets of Y. We wish to show that there is a sequence (of non-empty irreducible closed subsets) $X_0 \subsetneq ... \subset X_m$ of X such that $f(X_i) = Y_i$. Write $f^{-1}Y_m$ as a union of irreducible components $V_1 \cup ... \cup V_t$. Since f is surjective and finite, $Y_m = f(V_1) \cup ... \cup f(V_t)$, where $f(V_t)$ are closed and irreducible. Since Y_m is irreducible, we must have $Y_m = f(V_j)$ for some index j. By induction on m, we may find a chain of non-empty closed irreducibles $X_0 \subsetneq ... \subsetneq X_{m-1}$ of V_j with $f(X_i) = Y_i$. Then $X_0 \subsetneq ... \subsetneq X_{m-1} \subsetneq V_j$ is the desired sequence in X. □

Theorem 1.1. *$dim(\mathbb{A}^n) = n$*

Proof. $dim(\mathbb{A}^n) \geq n$ is clear. Suppose $Z_0 \subsetneq ... \subsetneq Z_m$ is a saturated chain of non-empty closed irreducible subsets of $\mathbb{A}^n$. We need to show that $m \leq n$. Then $Z_m = \mathbb{A}^n$ and Z_{m-1} is a closed, proper subset of $\mathbb{A}^n$. In particular, one can find a non-constant function $g \in k[X_1, ..., X_n]$ such that $Z_{m-1} \subseteq Z(g)$. By (the proof of) Noether normalization, there is a finite surjective morphism $Z(g) \to \mathbb{A}^{n-1}$. Then the previous lemma implies $dim(Z(g)) = dim(\mathbb{A}^{n-1})$. Inducting on n, we can assume $dim(\mathbb{A}^{n-1}) = n - 1$. Hence $m - 1 \leq dim(Z(g)) = dim(\mathbb{A}^{n-1}) = n - 1$, which completes the proof. □

Corollary 1. *If X is a hypersurface in $\mathbb{A}^n$ defined by a non-constant polynomial then $dim(X) = n - 1$.*

Corollary 2. *Every variety has finite dimension.*

We now return to curves.

Proposition 1. *All irreducible curves over a given field (or even various fields of equal cardinality!) are homeomorphic*

Proof. From the definition of dimension it is clear that a closed irreducible subset of an irreducible curve X is either zero dimensional or X. Any proper closed subset of X is therefore finite. Hence, any bijection between irreducible curves is a homeomorphism. But a curve over a field k has as many points as k. The proposition follows. □

Definition 1. *Let $X \subset \mathbb{A}^n$ be a hypersurface defined by a polynomial g. Write g as a sum of homogenous components $g = g_m + g_{m+1} + ...$ with $g_m \neq 0$. If $0 \in X$, the multiplicity of X at 0 is defined to be the natural number m. The multiplicity at $p \in X$ is the multiplicity at 0 after applying a linear change of coordinates mapping p to 0.*

Definition 2. *Let X, Y be two curves in $\mathbb{A}^2$ with no common component and (a, b) be an intersection point. If I_X and I_Y are the ideals in $k[x, y]$ defining X and Y, respectively. Then $V = k[x, y]/(I_X + I_Y)$ is a finite dimensional vector spaces and multiplication by x,y induce two commuting operators on V. The multiplicity of intersection of X and Y at (a, b) is defined as dimension of the common generalized eigenspace of the two operators, with eigenvalues a,b respectively.*

Theorem 1.2 (Bezout)**.** *Let $X, Y \subset \mathbb{P}^2$ be curves without a common component, of degree d and e, respectively. Then $X \cap Y$ contains de points, counted with multiplicities.*

Proof. Proof in lecture notes from 11/5. □

Theorem 1.3 (Pascal)**.** *Let Q be a circle in $\mathbb{P}^2$ and X a hexagon inscribed in C. Then the three pairs of opposite sides of X intersect at three points which lie on a straight line.*

Proof. Let A, B, C be linear equations of three pairwise nonintersecting sides of our hexagon inscribed in Q and A', B', C' be the equations of the remaining three ones with A' opposite to A etc. Pick a 7th point on Q and consider a degree 3 homogeneous polynomial P=ABC - t A'B'C' where t is such that P vanishes at the chosen 7th point. By Bezout's theorem, the intersection of Q with a deg 3 curve has at most 6 points, unless they have a common component. Since P has at least 7 zeroes, the latter must be true. Hence, the vanishing locus of P is the union of Q with some other component, which has to be a line L by a degree count. Now the intersection point of A and A' has to lie on L, as well as that of B with B' and C with C'. □

Theorem 1.4. *Let X be an irreducible variety of dimension n and let g be a non-constant function on X. Then any irreducible component of $Z(g)$ has dimension $n-1$.*

Lemma 3. $dim(Z(g)) \geq n-1$.

Proof. The special case $X = \mathbb{A}^n$ is proved above. We will reduce to this special case by Noether's lemma: choose $B = k[x_1, ..., x_n] \subset k[X] = A$ such that A is a finitely-generated B-module. Then g is the root of some monic irreducible polynomial $P \in B[t] = k[x_1, ..., x_n, t]$. Write $P = a_0 + a_1 t + ... + t^n$ with $a_i \in B$. The inclusion $B \subset A$ descends to a map $B/(a_0) \to A/(g)$. It is enough to show that the map of spectra $Spec(A/(g)) \to Spec(B/(a_0))$ is surjective. Let $C = B[t]/(P)$ and factor $B \subset A$ as $B \subset C \subset A$. $Spec(C)$ is irreducible of dimension n. Thus $\pi : Spec(A) \to Spec(C)$ is onto, so the preimage $\pi^{-1}(Z(t)) = Z(g)$ maps onto $Z(t)$. But $B/(a_0) \subset C/(t) = B/(\text{free terms of polynomials in } P)$. □

Lemma 4. *Let X be an irreducible variety and $U \subset X$ a non-empty open subset. Then $dim(U) = dim(X)$.*

Proof. If we replace X by $\mathbb{A}^n$ the lemma is clear: $dim(U) \leq dim(X)$ since $U \subseteq X$ and the chain (point in U) $\subsetneq$ line $\subsetneq ... \subsetneq \mathbb{A}^n$ of closed irreducibles in U shows that $dim(U) \geq dim(X)$. For X affine, use Noether's lemma to get a finite surjection $\pi : X \to \mathbb{A}^n$. Since π is closed, $V = \mathbb{A}^n - \pi(X - U)$ is open. Let $U' = \pi^{-1}V$. Then $\pi : U' \to V$ is a finite surjection. Hence, $dim(U') = dim(V) = n$. On the other hand, $U' \subseteq U$ so $dim(U') \leq dim(U) \leq dim(X) = n$. So $dim(U) = n$ as desired. For general X, reduce to the affine case by using $dim(X) = \max\{dim(U); U \text{ affine}\}$. □

Proof of Theorem 1.4. Assume Z is a component of $Z(g)$ and $dim(Z) \leq dim(X) - 2$. We can find an open affine subvariety U of X such that $U \cap Z(g) = Z \cap U$ is non-empty. Then by lemma 4 we have $dim(U \cap Z) = dim(Z) \leq dim(X) - 2 = dim(U) - 2$. Then by lemma 3, $g|_U$ is constant. But U is an open subset in an irreducible variety and therefore dense, so continuity implies g is globally constant. □

18.725 Algebraic Geometry
Fall 2015