Lecture 6: Function Field, Dominant Maps

Definition 1. *Let X be an irreducible variety. The function field of X, denoted $k(X)$ is defined as the limit*

$$K(X) = \lim_{U \subset X} k[U]$$

taken over all open subsets of X with the obvious restriction morphisms.

If X is irreducible, $k(X)$ is just the fraction field of the integral domain $k[U]$ for any open affine subset $U \subseteq X$. A morphism of varieties $f : X \to Y$ is dominant if the image of f is dense. Suppose $f : X \to Y$ is dominant and ϕ is a rational function on Y. Then by definition ϕ is an equivalence class $(U, g \in k[U])$, where (U, g) and (U', g') are equivalent if they restrict to the same function on an open subset of $U \cap U'$. Pick a representative (U, g) for ϕ. Since $f(X)$ is dense, $f^{-1}(U)$ is non-empty. Hence, $(f^{-1}(U), f^*g)$ is a rational function on X. It is easy to see that 'equivalent' functions on Y pull back to 'equivalent' functions on X. Thus, we obtain a map of function fields $f^* : k(Y) \to k(X)$.

Definition 2. *For any dominant map of irreducible varieties $f : X \to Y$ we obtain a field extension $k(X)/f^*k(Y)$. The degree of f is the degree of this field extension.*

Lemma 1. *Let X and Y be irreducible varieties with Y normal and $f : X \to Y$ a finite dominant map. Then for any $y \in Y$, $\#f^{-1}(y) \leq deg(f)$.*

Proof. Since f is finite (hence affine) we may reduce to the case where $X = Spec(A)$ and $Y = Spec(B)$. Finiteness implies that A is a finitely-generated B-module. Suppose $\#f^{-1}(y) = m$ and let $\phi \in A$ be a function taking distinct values on the elements of $f^{-1}(y)$. Let $P \in B[t]$ be the minimal polynomial for ϕ. Then $deg(P) \leq deg(f)$. Since Y is normal, B is integrally closed. Hence, the coefficients of P are elements of B and are therefore constant on $f^{-1}(y)$. Let $\tilde{P} \in k[t]$ denote the polynomial obtained from P by replacing the coefficients with their values at y. $\tilde{P}$ has at least m roots and hence $m \leq deg(\tilde{P}) = deg(P) \leq n$ which completes the proof. □

Definition 3. *Let X, Y be irreducible varieties, and let $f : X \to Y$ be a dominant map of degree n. f is* unramified *over $y \in Y$ if $\#f^{-1}(y) = n$. Otherwise, we say that f is* ramified *at y or that y is a* ramification point *of f.*

Proposition 1. *Let $f : X \to Y$ be a finite dominant map of irreducible varieties and let $R \subseteq Y$ be the set of ramification points. R is a closed subset of X and if the field extension $k(X)/f^*k(Y)$ is separable, then $R \neq X$.*

Proof. Since f is finite (hence affine), we may reduce to the case where X, Y are affine. We will first prove that $Y - R$ is open. Suppose f is unramified over y. Choose ϕ as in the proof of lemma 1. Since f is unramified at y, $\tilde{\phi}$ has n distinct roots, where $n = deg(f)$. Write $D(\phi)$ for the discriminant of f. $D(\tilde{\phi}) = D(\phi)(y) \neq 0$ implies f unramified at y. But $D(\phi)(y') \neq 0$ for y' in a neighborhood of y by continuity. Hence, $Y - R$ is open. Suppose $k(X)/f^*k(Y)$ is separable. Then $k(X)$ is generated over $f^*k(Y)$ by a single element $a \in A$ by field theory. Let F denote the minimal polynomial for a. Then $deg(F) = n$ and $D(F) \neq 0$ since F has no repeated roots. Hence, there are elements $y \in Y$ with $D(F)(y) \neq 0$. These will not be ramification points of f. □

We finish the lecture by stating an easy but extremely important general categorical result called Yoneda's Lemma. It says roughly that an object in a category is uniquely determined by a functor it represents. The standard way to apply it in algebraic geometry is as follows. Due to Yoneda's Lemma, to define an algebraic variety X, it suffices to describe the functor represented by X and then check that the functor is representable. This a standard tool used to make sense of the intuitive idea "the variety X parametrizing algebraic (or algebro-geometric) data of a given kind" – such as the Grassmannian variety parametrizing linear subspaces of a given dimension in k^n. More complicated examples (beyond the scope of 18.725) involve subvarieties in a given variety with fixed numerical invariants etc. In the next lecture we will use Yoneda Lemma to define products of algebraic varieties.

Lemma 2 (Yoneda)**.** *Let C be a category. For every $x \in C$ define a covariant functor*

$$\begin{aligned} h^x : & C \to Set \\ & c \mapsto Hom(x, c) \end{aligned}$$

Then the assignment $x \mapsto h^x$ defines a functor $h : C \to Functors(C, Set)$. h is fully faithful and therefore injective on objects (up to isomorphism).