# Lecture 7: Product of Varieties, Separatedness

Here are some additions to last time. Recall that if $R(X) \cong R(Y)$, then there are open subsets $U \subseteq X$, $V \subseteq Y$ which are isomorphic. To see this, replace $X, Y$ with $U, V$ such that we have morphisms $f : U \to V$ and $g : V' \to U$ (where $V' \subseteq V$) which are induced by the isomorphism $R(X) \cong R(Y)$. Then $fg : V' \to V'$ is the identity (induced by $R(Y) \to R(X) \to R(Y)$ which is the identity). Then $g : V' \to f^{-1}(V')$, and set $U' = f^{-1}(V')$. Then $gf : U' \to U'$ is the identity for similar reasons. Hence $U' \simeq V'$.

In the proof of a lemma from last time (that the set of unramified points is open), we used that if $\mathrm{Spec} A \to \mathrm{Spec}(C = B[t]/P) \to \mathrm{Spec} B$ (where everything has dimension $n$), then $C \subseteq A$; that is, $C \to A$ is an injection. If not, then the kernel is nontrivial, and consequently Spec(image) has dimension less than $n$, and hence $\dim \mathrm{Spec} A < n$.

**Products** Let $\mathcal{C}$ be any category and $X, Y \in \mathrm{Ob}(C)$. Then $X \times Y$ is an object $Z \in \mathrm{Ob}(C)$ together with maps $\pi_X : Z \to X$, $\pi_Y : Z \to Y$ such that for any other $T \in \mathrm{Ob}(C)$, there is an isomorphism $\mathrm{Hom}(T, Z) \xrightarrow{\sim} \mathrm{Hom}(T, X) \times \mathrm{Hom}(T, Y)$ given by $f \mapsto (\pi_X \circ f, \pi_Y \circ f)$. Equivalently, $X \times Y$ is the object corresponding to the functor $T \mapsto \mathrm{Hom}(T, X) \times \mathrm{Hom}(T, Y)$, if it exists. Yoneda's lemma implies that if it exists, then it is unique up to unique isomorphism.

Similarly, the coproduct $X \amalg Y$ is defined such that $\mathrm{Hom}(X \amalg Y, T) \xrightarrow{\sim} \mathrm{Hom}(X, T) \times \mathrm{Hom}(Y, T)$ .

**Example 1.** *Let $\mathcal{C}$ be the category of commutative $k$-algebras. Then the product is the usual direct product, or direct sum. The coproduct of $A, B$ would be $A \otimes_k B$. We have an equivalence of categories*

$$\{\text{affine algebraic varieties}\} = \{\text{finitely generated commutative nilpotent-free } k\text{-algebras}\}^{\mathrm{op}},$$

*where the* op *means the opposite category; the objects are the same, but the arrows are reversed. Thus, product of affine algebraic varieties corresponds to the tensor product of their global sections.*

**Exercise 1.** *Describe the product and coproduct in the category of not necessarily commutative $k$-algebras.*

**Lemma 1.** *If $A, B$ are nilpotent-free $k$-algebras, so is $A \otimes_k B$.*

*Proof.* We check that $A \otimes_k B$ injects into $\mathrm{Hom}_{k-\mathrm{alg}}(\mathrm{Spec} A \times \mathrm{Spec} B)$. For contradiction, take a nonzero element $\sum a_i \otimes b_i \in A \otimes_k B$ in the kernel. Without loss of generality, the $a_i$ are linearly independent, as well as the $b_i$. Find $x \in \mathrm{Spec} A$ such that for some $i$, $a_i(x) \neq 0$. Restricting to $\{x\} \times \mathrm{Spec} B$, we get a contradiction to linear independence of the $b_i$. Therefore, we can identify $A \otimes_k B$ with a subspace of $\mathrm{Hom}_{k-\mathrm{alg}}(\mathrm{Spec} A \times \mathrm{Spec} B)$, which clearly contains no nilpotents. □

Therefore, $\mathrm{Spec} A \otimes_k B$ makes sense, and $\mathrm{Hom}(X, \mathrm{Spec} A \otimes_k B) = \mathrm{Hom}(A \otimes_k B, k[X]) \simeq \mathrm{Hom}(A, k[X]) \times \mathrm{Hom}(B, k[X]) = \mathrm{Hom}(X, \mathrm{Spec} A) \times \mathrm{Hom}(X, \mathrm{Spec} B)$ implies that $\mathrm{Spec} A \times \mathrm{Spec} B = \mathrm{Spec} A \otimes_k B$.

**Remark 1.** *Caution: The topology on the product of spaces with functions is* ***not*** *the product topology.*

Suppose $X, Y$ are algebraic varieties, or spaces with functions. We define a basis of open sets on $X \times Y$ to be those subsets of the form $U \subseteq V_1 \times V_2$, where $V_1 \subseteq X$, $V_2 \subseteq Y$ are open and $U$ is the complement to zeroes($f = \sum f_i g_i$) where $f_i$ are regular on $V_1$, $g_i$ are regular on $V_2$. Another construction can be given as follows: suppose that $X$ and $Y$ can be written as $X = \cup U_i$, $Y = \cup V_j$ for $U_i = \mathrm{Spec} A_i$ and $V_j = \mathrm{Spec} B_j$. Then $X \times Y$ will be $\cup \mathrm{Spec}(A_i \otimes B_j)$, glued properly.

**Theorem 1.1.** $\dim(X \times Y) = \dim(X) + \dim(Y)$

*Proof.* The computation is local, so assume $X, Y$ are affine of dimension $n, m$ respectively. Then there are finite onto maps $X \twoheadrightarrow \mathbb{A}^n$, $Y \twoheadrightarrow \mathbb{A}^m$, so their product is a finite onto map $X \times Y \to \mathbb{A}^{n+m}$, which implies that $X \times Y$ is of dimension $n + m$. □

**Lemma 2.** *Suppose that for $i \in \{1, 2\}$, $X_i$ is a closed subvariety of $Y_i$. Then $X_1 \times X_2$ is a closed subvariety of $Y_1 \times Y_2$.*

*Proof.* Work locally to reduce to the case when $Y_1, Y_2$ are affine. The corresponding algebraic statement to check is that the tensor product of two surjective maps is still surjective; this is true. □

**Proposition 1.** *The product of projective varieties is projective.*

*Proof.* By the previous lemma, it suffices to check that $\mathbb{P}^n \times \mathbb{P}^m$ is projective. To do so, use the Segre embedding into $\mathbb{P}^{nm+n+m}$. Geometrically, the Segre embedding takes $(x, y) \in \mathbb{P}^n \times \mathbb{P}^m$, considers the duals of $x, y$ given by lines $L_x \subseteq k^{n+1} = V$, $L_y \subseteq k^{m+1} = W$, takes the line $L_x \otimes L_y \subseteq V \times W = k^{(n+1)(m+1)}$, and identifies that with its dual, which is a point in $\mathbb{P}^{nm+n+m}$. More concretely, it takes $((x_0 : \cdots : x_n), (y_0 : \cdots : y_m)) \mapsto (\cdots : x_i y_j : \cdots)$. If the coordinate are given by $z_{ij}$ such that the $x_i y_j$ belongs to the $z_{ij}$ coordinate, then the image of the Segre embedding is cut out by $z_{ij} z_{kl} - z_{kj} z_{il}$. □

**Separatedness**

**Example 2.** *Here is a non-quasiprojective variety: the line with a double point. It is given by* $\mathbb{A}^1 \times \{0, 1\}/((x, 0) \sim (x, 1) \text{unless} x = 0)$.

**Definition 1.** *An algebraic variety is separated if its diagonal* $\Delta_X$ *is a closed subvariety in* $X \times X$.

In general, the diagonal is always a locally closed subvariety. Furthermore, affine varieties are separated because if $X = \text{Spec}A$, then the multiplication map $A \otimes A \twoheadrightarrow A$ is surjective. Therefore, if $X$ is an algebraic variety such that $X = \cup U_i$ where the $U_i$ are affine, then $\Delta_X \cap (U_i \times U_i)$ is closed in $U_i$.

**Lemma 3.** *A locally closed subvariety in a separated variety is separated.*

*Proof.* Suppose $X$ is separated and $Z \subseteq X$ is a subvariety. Then $Z \times Z \subseteq X \times X$ is a subvariety, and $\Delta_Z = \Delta_X \times (Z \times Z)$. □

**Lemma 4.** $\mathbb{P}^n$ *is separated.*

*Proof.* Write $\mathbb{P}^n = \cup \mathbb{A}^n_i$. Then $\mathbb{A}^n_i \times \mathbb{A}^n_j \supseteq \Delta \cap (\mathbb{A}^n_i \times \mathbb{A}^n_j)$. When $i = j$, we are reduced to the affine case. When $i \neq j$, say $i = 0$ and $j = 1$, we take coordinates $x_1, \cdots, x_n$ and $y_0, y_2, \cdots, y_n$ and see that being on the diagonal is the closed condition $x_a y_b = x_b y_a$. □

**Corollary 1.** *A quasiprojective variety is separated.*

The line with a doubled origin is not separated. To see this, denote this algebraic variety by $X$, and note that we have a natural map $X \to \mathbb{A}^1$. Then $X^2 \to \mathbb{A}^2$, and over 0 we have $\{0_{ij}\}_{i,j \in \{1,2\}}$. The closure of diagonal contains all four points, while only two points $0_{11}$ and $0_{22}$ belong to the diagonal. In particular, $X$ cannot be quasiprojective as it is not separated.

**Remark 2.** *Often (including Hartshorne), an "abstract variety" is taken to be separated and irreducible.*

**Definition 2.** *Let* $f : X \to Y$ *be a morphism. Then* $\Gamma_f$*, called the graph of* $f$*, is the image of* $\text{id} \times f$ *in* $X \times Y$.

Note that $\Gamma_f$ is a subvariety isomorphic to $X$, and $\Gamma_{\text{id}}$ is the diagonal. Furthermore, $\Gamma_f$ is always locally closed. If $Y$ is separated, then $\Gamma$ is a closed subvariety.

**Corollary 2.** *If* $X$ *is irreducible and* $Y$ *is separated and* $f, g : X \to Y$ *agree on a nonempty open set, then* $f = g$.

*Proof.* Suppose $f, g$ agree on a nonempty open set $U \subseteq X$. Then $\Gamma_f|_U = \Gamma_g|_U$, and taking closures gives that $\Gamma_f = \overline{\Gamma_f|_U} = \overline{\Gamma_g|_U} = \Gamma_g$. Therefore, $f = g$. □

**Corollary 3.** *Suppose* $X$ *is irreducible,* $Y$ *is separated,* $U$ *is a nonempty open subset of* $X$*, and* $f : U \to Y$ *is a morphism. Then there is a maximal open subset* $V$ *of* $X$ *to which* $f$ *extends.*

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