Lecture 9: Chow's Lemma, Blowups

Last time we showed that projective varieties are complete. The following result from Wei-Liang Chow gives a partial converse. Recall that a birational morphism between two varieties is an isomorphism on some pair of open subsets.

Lemma 1 (Chow's Lemma)**.** *If X is a complete, irreducible variety, then there exists a projective variety $\tilde{X}$ that is birational to X.*

Proof. This proof is a standard one. Here we follow the proof presented by [SH77]. Choose an affine covering $X = U_1 \cup \ldots \cup U_n$, and let $Y_i \supseteq U_i$ be projective varieties containing U_i as open subsets. Now consider $\Delta : U \to U^n \to \prod_i U_i \to Y$ where $U = \bigcap_i U_i, Y = \prod_i Y_i$, and $\phi : U \to X \times Y$ be induced by the standard inclusion $U \to X$ and Δ. Let $\tilde{X}$ be the closure of $\phi(U)$, and π_1 gives a map $f : \tilde{X} \to X$. This map is birational because $f^{-1}(U) = \phi(U)$, and on U the map $\pi_1 \circ \phi$ is just identity. (To see the first claim, note that it means $(U \times Y) \cap \tilde{X} = \phi(U)$, i.e. $\phi(U)$ is closed in $U \times Y$, which is true because $\phi(U)$ in $U \times Y$ is just the graph of Δ, which is closed as Y is separated.)

So it remains to check that $\tilde{X}$ is projective. We show this by showing that the restriction of $\pi_2 : X \times Y \to Y$ to $\tilde{X}$, which we write as $g : \tilde{X} \to Y$, is a closed embedding. Let $V_i = p_i^{-1}(U_i)$, where p_i is the projection map from Y to Y_i. First we claim that $\pi_2^{-1}(V_i)$ cover $\tilde{X}$, which easily follow from the statement that $\pi_2^{-1}(V_i) = f^{-1}(U_i)$, since U_i cover X. Consider $W = f^{-1}(U) = \phi(U)$ as an open subset in $f^{-1}(U_i)$: on W we have $f = p_i g$, so the same holds on $f^{-1}(U_i)$ and the covering property follows.

It remains to show that $\tilde{X} \cap V_i \to U_i$ are closed embeddings. Noting that $V_i = Y_1 \times \ldots \times Y_{i-1} \times U_i \times Y_{i+1} \times \ldots \times Y_n$, we write Z_i to denote the graph of $V_i \xrightarrow{p_i} U_i \hookrightarrow X$, and note that it is closed and isomorphic to V_i via projection. Noting that $\phi(U) \subseteq Z_i$ and that Z_i is closed, taking closure we see that $\tilde{X} \cap V_i \to U_i$ is closed in Z_i. □

Blowing up of a point in $\mathbb{A}^n$ The blow-up of the affine n-space at the origin is defined as $\widehat{\mathbb{A}^n} = Bl_0(\mathbb{A}^n) \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1} = \{(x, L) : x \in \mathbb{A}^n, L \in \mathbb{P}^{n-1}, x \in L\}$. It is a variety defined by equations $x_i t_j = x_j t_i$. We have a projection $\pi : \widehat{\mathbb{A}^n} \to \mathbb{A}^n$. Atop 0 there is an entire $\mathbb{P}^{n-1}$, and on the remaining open set the projection is an isomorphism.

Now consider X an closed subset of $\mathbb{A}^n$, such that $\{0\}$ is not a component. The **proper transform** of X (a.k.a. the **blowup** of X at 0), denoted $\tilde{X}$, is the closure of the preimage of $X \setminus 0$ under π. Suppose X contains 0, then $\pi^{-1}(X) = \tilde{X} \cup \mathbb{P}^{n-1}$. If $X \subsetneq \mathbb{A}^n$, then $\mathbb{P}^{n-1} \not\subseteq \tilde{X}$ because $\dim(\mathbb{P}^{n-1}) \geq \dim(\tilde{X})$. If X is irreducible, then $\tilde{X}$ is the irreducible component of $\pi^{-1}(X)$ other than $\mathbb{P}^{n-1}$. The preimage of 0 within $\tilde{X}$ is called the **exceptional locus**.

Next, observe that $\widehat{\mathbb{A}^n}$ is covered by n affine charts. More explicitly, $\widehat{\mathbb{A}^n}_i \subseteq \mathbb{A}_i^{n-1} \times \mathbb{A}^n$ has coordinates $(t_1^i, \ldots, t_{i-1}^i, t_{i+1}^i, \ldots, t_n^i)$. On there, the defining equation becomes $x_j = t_j^i x_i$ for $j \neq i$, so $\widehat{\mathbb{A}^n}_i \cong \mathbb{A}^n$ with coordinates $(t_1^i, \ldots, t_{i-1}^i, x_i, t_{i+1}^i, \ldots, t_n^i)$. In other words, if $P(x_1, \ldots, x_n) \subseteq I_X$, then $P(t_1^i x_i, \ldots, t_{i-1}^i x_i, x_i, \ldots) \subseteq I_{\tilde{X} \cap \widehat{\mathbb{A}^n}_i}$.

Example 1. *Let $X = (y^2 = x^3 + x^2) \subseteq \mathbb{A}^n$. Suppose $y = tx$, then $t^2 x^2 = x^3 + x^2 \implies t^2 = x + 1$, so the preimage of $(0, 0)$ is $\{(t = \pm 1, x = 0)\}$. Thus X is not normal because the map $\tilde{X} \to X$ is not 1-to-1, though $\deg(\tilde{X} \to X) = 1$ (recall that a finite birational morphism to a normal variety is isomorphism).*

Definition 1. *Let X an affine variety, $x \in X$, we write $Bl_x(X) = \tilde{X}_x$ to denote $\tilde{X}$ for an embedding $X \subseteq \mathbb{A}^n$ where $x \mapsto 0$.*

Remark 1. *$Bl_x(X)$ contains $X \setminus x$ as an open set, so this generalizes to any variety X.*

Proposition 1. *Suppose X embeds via two embeddings i_1, i_2 to $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively, such that there exists some x such that $i_1(x) = i_2(x) = 0$, then $\tilde{X}_1 = \tilde{X}_2$ for two blowups at x.*

In particular, this tells us that blowup is an intrinsic operation that does not depend on the embedding.

Proof. First consider the special case $X = \mathbb{A}^n$, $i_1 = id$, and i_2 given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, f)$ for some polynomial f. Write $\widehat{\mathbb{A}^{n+1}} = \bigcup_{i=1}^{n+1} \mathbb{A}_i^{n+1}$, and observe that $\bigcup_{i=1}^{n} \mathbb{A}_i^{n+1} = \widehat{\mathbb{A}^{n+1}} \setminus \{(0 : 0 : \ldots : 0 : 1) \in \mathbb{P}^n\}$. Call that point ∞, then one can check that $\infty \notin \tilde{\mathbb{A}^n}$. Now note that $\tilde{\mathbb{A}^n} \cap \mathbb{A}_i^{n+1} \cong \mathbb{A}_i^n \subseteq \widehat{\mathbb{A}^n}$ (Locally write it as $t_{n+1}x_i = f(t_1x_i, \ldots, x_i, \ldots, t_nx_i)$, and observe we have a x_i on both sides so the closure would be of shape $t_{n+1} = f'(t_1, \ldots, x_i, \ldots, t_n)$, which gives an entire $\mathbb{A}^n$), so together we see that the blowup is nothing but $\widehat{\mathbb{A}^n}$. Second, consider $X = \mathbb{A}^n$, $i_1 = id$, $i_2 : \mathbb{A}^n \hookrightarrow \mathbb{A}^{n+m}$ being a graph of a morphism $\mathbb{A}^n \to \mathbb{A}^m$. This can be reduced to the first case by induction on m (or really, just the exactly same argument applied several times). Now consider the general case of arbitrary i_1, i_2. First extend the embedding $i_2 : X \to \mathbb{A}^m$ to a map $\mathbb{A}^n \to \mathbb{A}^m$ by lifting each generator (one can switch to the algebraic side, suppose $X = \operatorname{Spec} A$, then we get two surjective maps $\psi_1 : k[x_1, \ldots, x_m] \to A$ and $\psi_2 : k[y_2, \ldots, y_n] \to A$, lift ψ_1 to $\psi_2 \circ \phi$ for $\phi : k[x_1, \ldots, x_m] \to k[y_1, \ldots, y_n]$ where we map each x_i into A then lift), then one can use part 2. ($x \mapsto i_1(x) \mapsto i_1(x)$ has the same blowup as $x \mapsto i_1(x) \mapsto (i_1(x), i_2(x))$, which has the same blowup as $x \mapsto i_2(x) \mapsto i_2(x)$ by the same argument applied on the other direction.) □

As an application, consider an example of a complete non-projective surface: start with $\mathbb{P}^1 \times \mathbb{P}^1$, blow it up at $(0, 0)$, consider the projection to the second factor. For any $x \neq 0$, the preimage of x is a projective line; for $x = 0$, the preimage is the union of two projective lines (one can see this by passing to affine chart then consider closure). Consider two copies of this blow up, call them X, Y, and call the two exceptional lines L_1, L_2 for both of them, Now consider the disjoint union of X and Y where we identify L_1 of X with the fiber of ∞ of Y, and vise versa.

References

[SH77] Igor Rostislavovich Shafarevich and Kurt Augustus Hirsch. *Basic algebraic geometry*. Vol. 1. Springer, 1977.

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Fall 2015