Lecture 10: Sheaves, Invertible Sheaves on $\mathbb{P}^1$

In this lecture, definition of sheaves will be given. In particular, we will talk about invertible sheaves on $\mathbb{P}^1$.

Presheaves and Sheaves on Topological Spaces Let X be a topological space.

Definition 1. *A* presheaf of sets $\mathcal{F}$ *on the topological space* X *is an assignment for an open subset* $U \subset X$ *of a set* $\mathcal{F}(U)$ *and for a pair of open subsets* $V \subset U \subset X$ *of a so called* restriciton map $\phi_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ *such that the following axioms hold:*

1. *for each triple of open subsets* $W \subset V \subset U \subset X$ *the composition of the restriction maps* $\phi_W^V \circ \phi_V^U$ *is equal to the restriction* ϕ_W^U*;*
2. *for each open subset* $U \subset X$*, the restriction* ϕ_U^U *is equal to the identity map.*

Elements of the sets $\mathcal{F}(U)$ *are called* sections of the presheaf $\mathcal{F}$ over the open subset U.

Example 1. *Let* X *be a topological space. Then the assignment for an open subset* U *of the set of all functions on* U *defines a presheaf. The same for all continuous functions.*

Example 2. *Let* X *be a manifold. Then the assignment for an open subset* U *of the set of all smooth functions defines a presheaf. Analogously, one can define the presheaf of all holomorphic functions on a complex manifold.*

Definition 2. *A presheaf* $\mathcal{F}$ *on the topological space* X *is called a* sheaf *if the following is true for any (possibly infinite) open covering of an open subset* $U = \bigcup_\alpha U_\alpha$*:*

1. *for a collection of sections* $(s_\alpha) \in \prod_\alpha \mathcal{F}(U_\alpha)$*, if they coincide on intersections, that is* $s_\alpha|_\beta = s_\beta|_\alpha$*, then there exists a section* s *on* U *such that* $s|_\alpha = s_\alpha$*;*
2. *the map* $\prod_\alpha \phi_{U_\alpha}^U$ *is injective.*

Remark 1. *Note that the second property of the sheaf means that the section* s *from the first property is unique.*

Now we will introduce two essential constructions regarding presheaves and sheaves. Let X and Y be two topological spaces, and let $f : X \to Y$ be a continuous map.

Definition 3. *Let* $\mathcal{F}$ *be a presheaf on* X*. Then its* pushforward along f *is a presheaf* $f_*\mathcal{F}$ *on* Y*, and is defined on an open subset* $V \subset Y$ *as* $f_*\mathcal{F}(V) \overset{def}{=} \mathcal{F}(f^{-1}V)$.

Exercise 1. *Check that* $f_*\mathcal{F}$ *is indeed a preasheaf. Check that if* $\mathcal{F}$ *is a sheaf, then the pushforward* $f_*\mathcal{F}$ *is also a sheaf.*

Definition 4. *Let* $\mathcal{G}$ *be a presheaf on* Y*. Then its* pullback along f *is a presheaf* $f^*\mathcal{G}$ *on* X*, and is defined on an open subset* $U \subset X$ *as* $f^*\mathcal{G}(U) \overset{def}{=} \lim_{V \supset f(U)} \mathcal{G}(V)$.

Exercise 2. *Check that* $f^*\mathcal{G}$ *is a preasheaf.*

Note that the pullback of a sheaf is not generally a sheaf. However, the notion of the pullback of a sheaf does exist, and it is introduced using the so called sheafification, which will be discussed in the next lecture.

Remark 2. *Both pushforward and pullback constructions are functorial, that is if we also have a continuos map* $g : Y \to Z$*, then* $g_* \circ f_* = (g \circ f)_*$ *and* $f^* \circ g^* = (g \circ f)^*$.

Sheaves in Algebraic Geometry The situation with sheaves in algebraic geometry differs from the general case, because we want to endow our sets of sections with the structure of modules over regular functions. To make these words more rigorous, we first introduce the *structure sheaf* $\mathcal{O}_X$ of an algebraic variety X over $\mathbb{K}$. Recall that we have defined an algebraic variety as a certain space with functions, so secretly we have already introduced the structure sheaf in the very beginning of the course. Now we will just denote rings of regular functions over an open subset $U \subset X$ by $\mathcal{O}_X(U)$.

Exercise 3. *Check that $\mathcal{O}_X$ is a sheaf. Check that all restriction maps are ring homomorphisms. The latter means that $\mathcal{O}_X$ is a* sheaf of rings.

Definition 5. *Let $\mathcal{M}$ be a sheaf on X. We say that $\mathcal{M}$ is a* sheaf of $\mathcal{O}_X$-modules *if for any open subset $U \subset X$ the set $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-module, and all restriction maps commute with the ring action.*

Example 3. *The sheaf $\mathcal{O}_X$ considered as a module over itself is an example of a sheaf of $\mathcal{O}_X$-modules. We can define the direct sum $\mathcal{M} \oplus \mathcal{N}$ of two sheaves of modules as $\big(\mathcal{M} \oplus \mathcal{N}\big)(U) \stackrel{def}{=} \mathcal{M}(U) \oplus \mathcal{N}(U)$ with the obvious ring action. So we can also introduce the sheaves of modules $\mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X$. They are called* free sheaves.

Definition 6. *A* locally free sheaf $\mathcal{M}$ of rank n *on an algebraic variety X is a sheaf of $\mathcal{O}_X$-modules such that for some open cover of the variety $X = \bigcup U_i$, the restrictions $\mathcal{M}|_{U_i}$ are free sheaves on U_i of rank n, that is $\mathcal{M}|_{U_i} \cong (\mathcal{O}_X|_{U_i})^n$.*

Example 4. *Let p be a point in $\mathbb{P}^1$, then we can define the ideal sheaf $\mathcal{O}(-p)$ of this point as a certain subsheaf of the structure sheaf $\mathcal{O}$:*

$$\mathcal{O}(-p)(U) = \{f \in \mathcal{O}(U) \mid f(p) = 0\}.$$

This sheaf is locally free and of rank one.

More generally, we can define the ideal sheaf of any closed subvariety of an algebraic variety in the same way — as the sheaf whose sections are exactly those sections of the structure sheaf which vanish on the closed subset. Ideal sheaves need not be locally free.

Exercise 4. *An ideal sheaf is locally free if and only if it is principal.*

Operations of taking direct sum and tensor product of the sheaves take locally free sheaves to locally free sheaves.

We will see in the sequel that locally free sheaves of rank one form a group under the operation of tensor product, with identity being the structure sheaf. This group is called *Picard group*.

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