Lecture 11: Sheaf Functors and Quasi-coherent Sheaves

Recall that last time we defined a sheaf and a presheaf on a topological space, respectively denoted as $\mathbf{Sh}(X) \subseteq \mathbf{PreSh}(X)$. We'll work with sheaves of abelian groups on k-vector spaces. (Recall that $\mathcal{F}(X) \in \mathbf{PreSh}(X)$ if $F(U)$ is a k-vector space, and $\mathcal{F}(U)$ restricts to $\mathcal{F}(V)$ if $V \subseteq U$.)

Proposition 1. *Presheaf of abelian groups on k-vector space is an abelian category.*

Proof. If $\mathcal{F} \xrightarrow{f} G$, then $\ker(f)(U) = \ker(\mathcal{F}(U) \to \mathcal{F}(U))$, and same for cokernel. □

Note that $\mathbf{Sh}(X)$ is a full abelian subcategory. Now we introduce the sheafification functor: the embedding functor $\mathbf{Sh} \to \mathbf{PreSh}$ has a left adjoint, sending a presheaf $\mathcal{F}$ to its associated sheaf $\mathcal{F}^{\#}$. Recall that a presheaf is a sheaf if for all $U = \bigcup U_\alpha$, we have the exact sequence $0 \to \mathcal{F}(U) \to \prod_\alpha \mathcal{F}(U_\alpha) \to \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$. So we define $\mathcal{F}^{\#}(U) = \varinjlim_{U=\cup_\alpha U_\alpha} \ker(\prod \mathcal{F}(U_\alpha) \to \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta))$. Another description is via stalks: let $\mathcal{F}$ be a presheaf on X, $x \in X$, and define $\mathcal{F}_x = \varinjlim_{x\in U} \mathcal{F}(U)$. Then $\mathcal{F}^{\#}(U) = \{\sigma \in \prod_{x\in U} \mathcal{F}_x \mid \forall x \in U, \exists V \ni x \subseteq U, s \in \mathcal{F}(V), \text{s.t. } \{\sigma_y\}_{y\in V} \text{ comes from } s\}$. This shows in particular colimits exist in $\mathbf{Sh}(X)$: $\text{coker}_{\mathbf{Sh}}(\mathcal{F} \to \mathcal{G}) = \text{coker}_{\mathbf{Presh}}(\mathcal{F} \to \mathcal{G})^{\#}$. This just follows from general abstract nonsense.

Example 1. *An example of a cokernel in **Presh** that is not a sheaf: take $X = S^1$, let $\mathcal{F}$ be the continuous function sheaf $C(X, \mathbb{R})$ (i.e. $\mathcal{F}(U)$ are the continuous maps $U \to \mathbb{R}$), and $\mathcal{G}$ be the constant sheaf $\mathbb{Z}$ (i.e. $\mathcal{G}(U)$ consists of constant $\mathbb{Z}$-valued function on each connected U; more precisely, $\mathcal{G}(U)$ are continuous maps $U \to \mathbb{Z}$ where the latter has the discrete topology), then $(\mathcal{F}/\mathcal{G})_{\mathbf{Sh}}(U)$ would be continuous maps $U \to \mathbb{R}/\mathbb{Z}$, whereas $(\mathcal{F}/\mathcal{G})_{\mathbf{Presh}}(U)$ would be the continuous maps $(U, \mathbb{R})$ mod out the constant maps.*

Proposition 2. *Some properties:*

1. *$\mathcal{F} \to \mathcal{F}^{\#}$ is exact; in particular it doesn't change the stalks.*
2. *$\mathcal{F} \to \mathcal{F}^{\#}$ is left adjoint to the embedding **Presh** $\to$ **Sh**, and is an isomorphism if $\mathcal{F}$ itself is a sheaf.*

   As an example, consider the constant presheaf $\underline{V}$ given by $\mathcal{F}(U) = V$ constant. Then $\mathcal{F}^{\#}$ is a constant sheaf given by $\mathcal{F}^{\#}(U) = \{\text{locallyconstantmaps } U \to V\}$. (Why is $\mathcal{F}$ not a sheaf itself? Answer: it fails the local identity axiom on $U = \emptyset$.)
3. *$\mathcal{F} \mapsto \mathcal{F}_x$ is an exact functor; in other words, a sequence of sheaves $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ is exact iff $0 \to \mathcal{F}_x \to \mathcal{F}'_x \to \mathcal{F}''_x \to 0$ is exact for all x.*

Pullback and Pushforward If $f : X \to Y$ is a continuous map, then we have $f^* : \mathbf{Sh}(Y) \to \mathbf{Sh}(X)$, and $f_* : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. The latter (pushforward) is given by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$, and the former (pullback) is given by the sheafification of the presheaf $\varinjlim_{f(U)\subseteq V} \mathcal{F}(V)$. In particular, we have $\mathcal{F}_x = i_x^*(\mathcal{F})$; so $f^*(\mathcal{F})_x = \mathcal{F}_{f(x)}$, and in particular, we see that f^* is exact. On the other hand, f_* is only left exact (to see it is not necessarily exact, note that the pushforward to a point is the same as the global section, which is not necessarily exact).

Structure Sheaf Suppose X is a space with functions, then X carries the structure sheaf $\mathcal{O}_X$, given by $\mathcal{O}_X(U) = k[U]$. Say $X = \text{Spec}(A)$ is affine, and $x \in X$, then $\mathcal{O}(X)_x$ is the localization of A at the maximal ideal $\mathfrak{m}_x$. This makes X a *ringed space*, i.e. a topological space equipped with a sheaf of rings.

A sheaf of modules over a ringed space (X, A) is a sheaf $\mathcal{F}$ where $\mathcal{F}(U)$ is an $A(U)$ module, such that the restriction to subsets respects the module structure. A sheaf of modules $\mathcal{F}$ on a ringed space (X, A) is *quasicoherent*, if $\forall x \exists U \ni x$ such that there exists an exact sequence $A_U^{\oplus I} \to A_U^{\oplus J} \to \mathcal{F}_U \to 0$, where the first two are free modules (with possibly infinite dimensions).

Remark 1. *Caution:* $\bigoplus_{j\in J} A$ *is the sum in the category of sheaves, given by* $(\bigoplus_{\textbf{\textit{PreSh}}} A)^{\#} = \{s \in \prod_{j\in J} A(U) \mid$ locally $s \in \bigoplus\}$, *i.e.* $\forall x \in U \exists V \ni x, V \subseteq U$ *such that only finitely many components of* $s|_V$ *are nonzero. One can check that the section matches with the normal notion of* $\bigoplus_J A(U)$ *if* U *is quasicompact. If* X *is Noetherian, then any open* U *is quasicompact, so* $(A^{\oplus J})(U) = A(U)^{\oplus J}$.

Lemma 1. *If* X *is Noetherian, then* $\Gamma(\varinjlim \mathcal{F})(U) = \varinjlim \mathcal{F}(U)$, *where the right side is the filtered direct limit.*

In general, if X is a topological space, Γ is the global section functor $\mathbf{Sh}(X) \to \mathbf{Vect}_k$, then it has a left adjoint $L(\Gamma)$ where $L(\Gamma)(V)$ the locally constant sheaf with values in V.

Quasicoherent $\mathcal{O}$-modules We denote the category of quasicoherent O_X modules by $\mathbf{QCoh}(X)$, where X is an algebraic variety.

Theorem 1.1. *If* $X = \mathrm{Spec}(A)$, *then* $\boldsymbol{QCoh}(X) \cong \boldsymbol{Mod}(A)$, *given by* $\mathcal{F} \to \Gamma(\mathcal{F}) = \mathcal{F}(X)$.

Proof. First construct the adjoint (localization) functor Loc, where we use $\tilde{M}$ to denote $\mathrm{Loc}(M)$. To do so, first construct a presheaf L that sends U to $k[U] \otimes_A M$, then sheafify this presheaf. The functor L is left adjoint to the canonical functor $\mathbf{Mod}(k[U]) \to \mathbf{Mod}(A)$, then one can deduce that L is left adjoint to Γ, which sends presheaves of $\mathcal{O}$-modules to A-modules, from which the theorem follows. □

Note that Loc is an exact functor, which follows from the description of the stalks. Note that $\mathcal{F}^{\#}$ is defined by $\mathcal{F}(U)$, where U is an fixed base of topology. In particular, use the base $\{U_f = X - Z_f\}$ (the Zariski topology), and note that $k[U_f] = A_{(f)}$, thus $k[U_f] \otimes_A M = M_{(f)}$, and note that $M \mapsto M_{(f)}$ is exact. Finally, $\tilde{M}_x = \varinjlim_{f | f(x) \neq 0} M_{(f)} = M_{\mathfrak{m}_x}$ is exact. It's clear that $\tilde{A} = \mathcal{O}$. As a corollary,

Corollary 1. $\tilde{M}$ *is a quasicoherent* O_X *module.*

To see this, choose a presentation, and observe that $\widetilde{\oplus_{i\in I} M_i} = \oplus \tilde{M}_i$.