Lecture 12: Quasi-coherent and Coherent Sheaves

We finish the proof of the following statement:

Theorem 1.1. *Let $X = \operatorname{Spec}(A)$ be an affine variety. Then there is an equivalence of categories f : $\boldsymbol{QCoh}(X) \cong \boldsymbol{Mod}(A)$.*

Proof. Last time we defined the left adjoint functor $\operatorname{Loc} : M \to \tilde{M}$, where the latter is the sheaf assigned to the presheaf $\mathcal{F}(U) = k[U] \otimes_A M$. Note that it is an exact functor. We have a natural functor $\mathbf{Mod}(A) \to \mathbf{Sh}(X) \to \mathbf{QCoh}(X)$.

Lemma 1. *Let $i \in I$ be a directed system indexing sheaves $\mathcal{F}_i$. If X is a Noetherian topological space, then $\varinjlim_{\boldsymbol{PreSh}} \mathcal{F}_i$ is a sheaf. Hence $\varinjlim_{\boldsymbol{PreSh}} \mathcal{F}_i = \varinjlim_{\boldsymbol{Sh}} \mathcal{F}_i$. (Note that $\varinjlim_{\boldsymbol{PreSh}} (\mathcal{F}_i)(U) = \varinjlim \mathcal{F}_i(U)$ whereas $\varinjlim_{\boldsymbol{Sh}}(\mathcal{F}_i) = \varinjlim(\mathcal{F}_i)^{\#}$.) This shows that $\varinjlim_{\boldsymbol{Sh}} \mathcal{F}_i(U) = \varinjlim(\mathcal{F}_i(U))$.*

Example 1. *Take $X = \mathbb{Z}$, then $\Gamma(\bigoplus k_n)$ (where k_n is supported at n) $= \prod_n k \supsetneq \bigoplus_n k_n$.*

Back to the proof of the theorem. We need to check that the sheaf condition holds for $U = \bigcup_\alpha U_\alpha$. U can be made quasicompact since we're Noetherian, so enough to consider the case where $\{U_\alpha\}$ is finite. Using induction we can reduce to $U = U_1 \cup U_2$. Now observe the following sequence is exact:

$$0 \to \varinjlim \mathcal{F}_i(U) \to \varinjlim F(U_1) \oplus \varinjlim F(U_2) \to \varinjlim F(U_1 \cap U_2)$$

Now suppose X is an algebraic variety. $U = U_f = X \setminus Z_f$, and $\mathcal{F}$ is quasicoherent.

Proposition 1. *$j_* j^* \mathcal{F} = \varinjlim(f^{-n}\mathcal{F})$, where $j : U \hookrightarrow X$, $j_*\mathcal{F}$ means the sheaf whose section on V is $\mathcal{F}(U \cap V)$, and the right side is the formal notation denoting copies of $\mathcal{F}$, where $\{f^{-n}\mathcal{F}, n = 0, 1, \ldots\}$ are combined in a direct system, and we have the mapping*

$$\mathcal{F} \xrightarrow{f} f^{-1}\mathcal{F} \xrightarrow{f} f^{-2}\mathcal{F} \xrightarrow{f} \ldots$$

Proof. From each $f^{-n}\mathcal{F}$ there is an obvious map $f^{-n}\mathcal{F} \to j_*j^*\mathcal{F}$ and thereby there is an induced map $\varinjlim f^{-n}\mathcal{F} \to j_*j^*\mathcal{F}$, which we want to show is an isomorphism. Suffices to assume X is affine. Recall that taking direct limit in presheaves and sheaves yield the same result for Noetherian spaces; in other words, for each U we have $(\varinjlim f^{-n}\mathcal{F})(U) = \varinjlim(f^{-n}\mathcal{F}(U))$, so it suffices to check that $\Gamma(X, j_*j^*\mathcal{F}) = \Gamma(X, j^*\mathcal{F}) = \varinjlim(f^{-n}\mathcal{F}(X))$, which holds because if $\Gamma(X, \mathcal{F}) = M$, then $\Gamma(X, j^*\mathcal{F}) = M_f = \varinjlim f^{-n}M = \varinjlim(f^{-n}\mathcal{F}(X))$. □

We'll write this limit as $\mathcal{F}_f$. To finish the proof, let us first check that $\Gamma : \mathbf{QCoh}(X) \to \mathbf{Mod}(A)$ is exact (Proposition II.5.6 of Hartshorne). Assuming X is separated, this is in fact true *if and only if* X is affine; this is known as **Serre's criterion**. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ and let $\sigma \in \Gamma(\mathcal{F}'')$, First, check for any $x \in X$ there exists $f \in A$ such that $f(x) \neq 0, f^n\sigma \in \operatorname{Im}(\Gamma(\mathcal{F}))$. By the exactness of the short exact sequence, $\exists U = U_f \ni x, \tilde{\sigma} \in \mathcal{F}(U), \tilde{\sigma} \to \sigma|_U$. Let $s/f^n = \tilde{\sigma} \in \Gamma(\mathcal{F})_f = \mathcal{F}(U)$, where $s \in \Gamma(\mathcal{F})$, then it goes into $\Gamma(\mathcal{F}'')_f = \mathcal{F}''(U)$. $s \mapsto f^n\sigma$ is the localized map, so $f^m s \mapsto f^{n+m}\sigma$ under the map $\Gamma(\mathcal{F}) \to \Gamma(\mathcal{F}'')$. Now let $s \in$ Coker, By what we just said, we can cover X by open sets U_{f_i} such that $f_i^n s = 0 \in$ Coker. Thus since f_i together generate 1, $s = 0$. So indeed it is onto.

Now we know $\Gamma(\tilde{A}) = A$. Loc commute with $\bigoplus \Gamma(\widetilde{A^{\oplus I}}) = A^{\oplus I}$. Given $M \in \mathbf{Mod}(A)$, take some presentation $A^{\oplus J} \to A^{\oplus I} \to M \to 0$, then the canonical map $\Gamma(\tilde{M}) \to M$ is an isomorphism. Now we need to check that $\widetilde{\Gamma(\mathcal{F})} \to \mathcal{F}$ is also an isomorphism. (The rest follows [Har77] as the proof in class was not recorded.) Quasicoherence of $\mathcal{F}$ means that there exists some open covering $X = \bigcup D(g_i)$ such that $\mathcal{F}|_{D(g_i)} = \tilde{M}_i$ for some modules (M_i). On the other hand, by Lemma 5.3 of [Har77], applied to $D(g_i)$, gives that $\mathcal{F}(D(g_i)) = \Gamma(\mathcal{F})_{g_i}$ (the localized module), so in fact we have $M_i = \Gamma(\mathcal{F})_{g_i}$ (as one can check on stalks), and thus $\widetilde{\Gamma(\mathcal{F})} \to \mathcal{F}$ is isomorphism on each $D(g_i)$, hence overall an isomorphism. □

A sheaf $\mathcal{F} \in \mathbf{QCoh}(X)$ is *coherent* if locally we have a s.e.s. $O_U^{\oplus I} \to O_U^{\oplus J} \to \mathcal{F} \to 0$, with I, J finite.

Lemma 2. *If $X = \text{Spec } A$, then $\mathcal{F} = \tilde{M}$ is coherent iff M is finitely generated.*

Proof. If M is finitely generated we clearly have a coherent sheaf. On the other hand, Suppose $\tilde{M}$ is coherent, then take an open cover of X by $D(f_i)$ such that on each $D(f_i)$, the restriction (which we denote by $\tilde{M_i}$) is a finitely-generated $k[X]_{f_i}$-module. Now observe that $\tilde{M_i} = M_{(f_i)}$, and since there are only finitely many f_i, after clearing the denominators we can get a finite generating set for M. □

Let $f : X \to Y$ morphism of algebraic varieties. For $\mathcal{F} \in \mathbf{Sh}_{O-\text{mod}}(X)$, we can define $f_*F \in \mathbf{Sh}_{O-\text{mod}}(Y)$ (pushforward or direct image) by $f_*(F)(U) = F(f^{-1}(U))$.

Lemma 3. *f_* sends $\textbf{\textit{QCoh}}(X)$ to $\textbf{\textit{QCoh}}(Y)$. Note that it does not send coherent module to coherent module. e.g. $f : \mathbb{A}^1 \to *$.*

Proof. First consider when X, Y affine. This becomes $\text{Spec}(A) \to \text{Spec}(B)$, $f_*(\tilde{M}) = \tilde{M_B}$ clear by inspection. Now for general X, Y, we can assume Y affine since the question is local. Let $X = \bigcup U_i$ and denote $U_i \cap U_j = \bigcup_k U_{ij}^k$, then there is an exact sequence

$$0 \to f_*(\mathcal{F}) \to \bigoplus_i (f|_{U_i})_*(\mathcal{F}|_{U_i}) \to \bigoplus_{i,j,k} (f|_{U_{ij}^k})_*(\mathcal{F}|_{U_{ij}^k})$$

Now apply Proposition II.5.7 of Hartshorne. □

Corollary 1. *f_* is exact for a map of affine varieties. It is left exact in general.*

We claim tha f_* has the left adjoint functor $f^* : \mathbf{QCoh}(Y) \to \mathbf{QCoh}(X)$. Recall that $M \mapsto M_B$ has left adjoint $M \mapsto A \otimes_B M$. This defines f^* for a map of affine varieties. In general, $f^*(F) = [O_X \otimes_{f^*(O_Y)} f^*(F)]^{\#}$.

General property about pullback: suppose $X \to Y$, $U = \text{Spec}(A)$ in X and $V = \text{Spec}(B)$ in Y. Let $F|_V = \tilde{M}$, then $f^*(F)|_U = \widetilde{A \otimes_B M}$. We see that f^*U is right exact by adjointness (or from the fact that tensor products are right adjoint).

A particular example of this is the pullback to a point. Consider $i : \{x\} = * \hookrightarrow X$. Then $i^*(\mathcal{F})$ is the fiber of $\mathcal{F}$ at x. If X is just quasicoherent, it may have zero fibers at points. (Consider the example $X = \mathbb{A}^1$, and $j : \mathbb{A}^1 - \{0\}$, and let $\mathcal{F} = j_*O/\mathcal{O}$, let $\tilde{M} = \mathcal{F}$, where $M = \dfrac{k[t, t^{-1}]}{k[t]} = \{a_{-1}t^{-1} + \ldots + a_n t^{-n}\}$, then the multiplication by t is surjective. What is the fiber of $\mathcal{F}$ at 0? it is $M/tM = 0$.) Also $\mathcal{F}|_{\mathbb{A}^1 - \{0\}} = 0$, so fiber at $x \neq 0$ is also 0.

Lemma 4. *If $\mathcal{F}$ is coherent, then:*

1. *Fiber is always finite dimensional;*
2. *Fiber of $\mathcal{F}$ at x is zero iff $\exists U \supseteq x, F|_U = 0$;*
3. *The function $d : x \mapsto \dim(\text{fiber}(x))$ is (upper) semicontinuous.*
4. *The function d is locally constant if and only if F is locally free.*

Proof. Part 1) is obvious. Now denote the fiber by $F_x(\mathcal{F})$. Let I_x be the stalk, i.e. module over the stalk of O, i.e. $O_{x,X}$-local ring of x. The claim is that $F_x(\mathcal{F}) = F_x/\mathfrak{m}_x I_x = I_x \otimes_{O_{x,X}} k$. Let $\overline{m_1}, \ldots, \overline{m_n}$ be a basis in $F_x(\mathcal{F})$, use Nakayama to find some $m_i \in F_{x_i}$ such that m_i generate F_x. So $F_x(\mathcal{F}) = 0 \implies F_x = 0 \implies F|_U = 0$ for some $U \ni x$. This finishes part 2). Now, $\exists U_i$ and action $s_i \in F(U) \mapsto m_i$, s_i generate $F(U)$ as $k(U)$ module. This is part 3). Part 4) is left as exercise. □

References

[Har77] Robin Hartshorne. *Algebraic geometry*. Vol. 52. Springer Science & Business Media, 1977.

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