Lecture 13: Invertible Sheaves

Last time we showed that when $X = \text{Spec } A$ is an affine scheme, we have the equivalence $\text{QCoh}(X) \cong \textbf{Mod}(A)$ given by the Γ and the Loc functors. In particular, these functors are exact, and we have $\Gamma(\mathcal{F}) = 0 \implies \mathcal{F} = 0$. This in particular implies that $\Gamma \circ \text{Loc} = 1$ (We know this holds for A, now check the general case by choosing a presentation.). We need to check the other direction: $\text{Loc} \circ \Gamma(\mathcal{F}) = \mathcal{F}$.

Definition 1. *A functor $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ is called conservative if for every $g \in \text{Hom}(\mathcal{C}_1)$, $\mathcal{F}(g)$ is an isomorphism implies that g is an isomorphism. Note that this does not say that $\mathcal{F}(A) \cong \mathcal{F}(B) \implies A \cong B$.*

Example 1. *Let $\mathcal{C}_1, \mathcal{C}_2$ be abelian categories, and $\mathcal{F}$ an exact functor. Then $\ker(\mathcal{F}(f)) = \mathcal{F}(\ker(f))$, and the same holds for cokernels.*

Lemma 1. *Let $\mathcal{L}$, $\mathcal{R}$ be adjoint functors, $\mathcal{L}$ fully faithful (i.e. $\mathcal{R} \circ \mathcal{L} \cong Id$), $\mathcal{R}$ is conservative, then the two functors are inverse pairs in an categorical equivalence.*

Proof. We need $\mathcal{RL} \cong Id$, which follows from $\mathcal{RLR} \cong \mathcal{R}$ by conservative property, which in turns follows from the fully faithfulness of $\mathcal{F}$. □

Now back to the discussion on Loc and Γ. We already know that Loc is fully faithful, and it is sufficient to show it is essentially surjective, i.e. every $\mathcal{F}$ has some M such that $\mathcal{F} = \widetilde{M}$. The image of $\widetilde{M}$ are the functors that have presentations, i.e. $\mathcal{O}^{\oplus I} \to \mathcal{O}^{\oplus J} \to \mathcal{F} \to 0$, so it suffices to check that every $\mathcal{F}$ has a presentation. We check that for every $\mathcal{F}$, there exists a surjection $\mathcal{O}^{\oplus J} \twoheadrightarrow \mathcal{F}$. To see so, consider $\Gamma(\mathcal{F}) = \text{Hom}(\mathcal{O}, \mathcal{F})$ (structure sheaf is the terminal object in the category of sheaves). So if we take a set of generators $m_j, j \in J$ of $\mathcal{F}$, we obtain an onto map $\Gamma(\mathcal{O}^{\oplus J}) \to \Gamma(\mathcal{F})$, so $\mathcal{O}^{\oplus J} \to \mathcal{F}$ is surjective.

Remark 1. *Results of this type are generally referred to as Morita theories.*

Now suppose A contains arbitrary direct sums and that $\text{Hom}(P, \bullet)$ commutes with the direct sum. We say $P \in A$ is a *projective generator* if the P-projection functor, $X \mapsto \text{Hom}(P, X)$, is an exact functor, and that $\text{Hom}(P, X) = 0 \Leftrightarrow X = 0$. In this case, one can show that $A \cong \textbf{Mod}(\text{End } P)^{opp}$, and, in particular, as a corollary, we have $\textbf{Mod}(A)_{f.g.} \cong \textbf{Coh}(X)$.

Lemma 2. *$f : X \to Y$ is an affine morphism if and only if for every open $U \subseteq U$, $f^{-1}(U)$ is affine. $f : X \to Y$ is a finite morphism if and only if it is affine and, for every open $U \subseteq Y$ such that $U = \text{Spec } A$, if $f^{-1}(U) = \text{Spec } B$ then B is a finite A-algebra.*

Proof. Let U be affine. By definition, there exists some affine cover $U = \bigcup U_i$ such that $f^{-1}(U_i)$ is affine. Write $V = f^{-1}(U)$, then we want to have $V = \text{Spec } A$. Note that $k[U_i] = f_*(\mathcal{O})(U_{f_i}) = f_*(\mathcal{O})(U)_{f_i} = A_{(f_i)}$, and each $A_{(f_i)}$ is finitely generated. Take all those rings together as an algebra over $B = k[U]$, we obtain a finitely generated ring A. The check that $V = \text{Spec } A$ is routine. For the second part, suppose $f : X \to Y$ finite (in the old definition), then $f_*\mathcal{O}_X$ is a coherent sheaf on Y, i.e. $f_*\mathcal{O}_X(U)$ is finite over $\mathcal{O}_Y$ for some open set U. □

Proposition 1. *For any fixed Y, the category of X that has an affine morphism to Y corresponds to the opposite category of quasicoherent sheaves of $\mathcal{O}_Y$-algebra (which is finitely generated and reduced).*

To see this, given any map $f : X \to Y$ we obviously obtain a sheaf $f_*\mathcal{O}_X$. Conversely, given a sheaf $\mathcal{A}$ of $\mathcal{O}_Y$ algebra, pick an affine cover $Y = \bigcup_i U_i$, glue together all the Spec $\mathcal{A}[U_i]$ by identifying Spec $\mathcal{A}[U_i \cap U_j]$ that sits in two copies (here we assume seperatedness).

Proposition 2. *Suppose $X \to Y$ is affine. Let $\mathcal{A} = f_*\mathcal{O}_X$, then **Qcoh**$(X) = \{$**Qcoh**$(Y)$ with an $\mathcal{A}$ action$\}$, where the map is $\mathcal{F} \mapsto f_*\mathcal{F}$.*

Let $i : Z \hookrightarrow X$ be an embedding of a closed subvariety, then i_* is a full embedding of a subcategory, with one-sided inverse i^*. It is easy to see that the image of i_* consists of those $\mathcal{F}$ such that $\mathcal{F}|_{X-Z} = 0$. On the other hand, for every $Z \subseteq X$ we have a subsheaf $\mathcal{I}_Z \subseteq \mathcal{O}_X$ consisting of those f that vanish on Z. It is obviously an ideal sheaf, and we in fact have a correspondence between closed subvarieties and radical ideal sheaves.

Proposition 3. *$i_*: \textbf{\textit{Qcoh}}(Z) \to \textbf{\textit{Qcoh}}(X)$ (or coherent to coherent) is a full embedding and the image are the $\mathcal{F}$s such that $\mathcal{I}_Z\mathcal{F} = 0$.*

For example, consider $X = \text{Spec } A$, and let $Z = \text{Spec } A/I$, then A/I modules are the A modules that are killed by I. Let $U = X - Z$, then $i_*\mathcal{F}|_U = 0$. Note the converse doesn't hold: there might be $\mathcal{F}$ that restricts to U to be trivial, but does not come from i_*M for any M. For instance, let $X = \mathbb{A}^1, Z = \{0\}$, let $M = k[t]/t^2$, $\mathcal{F} = \widetilde{M}$, and let $i : k[t] \to k$ that sends t to 0. There does exist a weaker property: if $\mathcal{F}|_U = 0$, σ is a section of $\mathcal{F}$, then there exists some n such that $\mathcal{I}_Z^n\sigma = 0$. In addition, if $\mathcal{F}$ is coherent, then we actually have ssome n such that $\mathcal{I}_Z^n\mathcal{F} = 0$.

Locally free sheaves of rank 1 are called **invertible sheaves**.

Example 2. *Let $X = \mathbb{P}^n$, then $\mathcal{O}_{\mathbb{P}^n}(d)(U) = k[\tilde{U}]_d = \{p/q \mid \deg p - \deg q = d, q|_{\tilde{U}} \neq 0\}$ is an invertible sheaf on X, where $\tilde{U} \hookrightarrow U$ is the projection compatible with $\mathbb{A}^{n+1} - \{0\} \hookrightarrow \mathbb{A}^{n+1}$.*

We would like to understand maps $X \to \mathbb{P}^n$, by which we mean the similar knowledge as the fact that T.F.A.E.:

- Maps $X \to \mathbb{A}^n$;
- Homs $k[x_1, \dots, x_n] \to k[X]$;
- n-tuple elements in $k[X]$.

And our claim is that T.F.A.E.:

- Maps $X \to \mathbb{P}^n$;
- Invertible sheaves $\mathcal{L}$ on X with $(n+1)$ elements $s_0, \dots, s_n$ in $\Gamma(\mathcal{L})$ such that they generate $\mathcal{L}$.

Here to a map $f : X \to \mathbb{P}^n$ we assign $f^*\mathcal{O}(1)$ with sections $t_0, \dots, t_n$. Conversely, given $\mathcal{L}$ generated by $s_0, \dots, s_n$ set $f = (s_0 : \dots : s_n)$, locally we can identify $\mathcal{L}$ with $\mathcal{O}$ so $s_0, \dots, s_n$ give functions on U with no common zeroes. If $f_0, \dots, f_n$ are these functions, then $x \mapsto (f_0(x) : \dots : f_n(x))$ is a map $U \mapsto \mathbb{P}^n$ indepdendent of choice that gives an isomorphism $\mathcal{L} \cong \mathcal{O}$.

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