Lecture 15: Divisors and the Picard Group

Suppose X is irreducible. The (Weil) divisor $\mathrm{Div}_W(X)$ is defined as the formal $\mathbb{Z}$ combinations of subvarieties of codimension 1. On the other hand, the Cartier divisor group, $\mathrm{Div}_C(X)$, consists of subvariety locally given by a nonzero rational function defined up to multiplication by a nonvanishing function.

Definition 1. *An element of* $\mathrm{Div}_C(X)$ *is given by*

1. *a covering* U_i*; and*
2. *Rational functions* f_i *on* U_i*,* $f_i \neq 0$*,*

such that on $U_i \cap U_j$*,* $f_j = \varphi_{ij} f_i$*, where* $\varphi_{ij} \in O^*(U_i \cap U_j)$*.*

Another way to express this is that $\mathrm{Div}_C(X) = \Gamma(K^*/\mathcal{O}^*)$, where K^* is the sheaf of nonzero rational functions, where $\mathcal{O}^*$ is the sheaf of regular functions.

Remark 1. *Cartier divisors and invertible sheaves are equivalent (categorically). Given* $D \in \mathrm{Div}_C(X)$*, then we get an invertible subsheaf in* K*, locally it's* $f_i\mathcal{O}$*, the* $\mathcal{O}$*-submodule generated by* f_i *by construction it is locally isomorphic to* $\mathcal{O}$*. Conversely if* $L \subseteq K$ *is locally isomorphic to* $\mathcal{O}$*, A system of local generators defines the data as above. Note that the abelian group structure on* $\Gamma(\mathcal{K}^*/\mathcal{O}^*)$ *corresponds to multiplying by the ideals.*

Proposition 1. $\mathrm{Pic}(X) = \mathrm{Div}_C(X)/\mathrm{Im}(\mathcal{K}^*) = \Gamma(\mathcal{K}^*/\mathcal{O}^*)/\operatorname{im}\Gamma(\mathcal{K}^*)$.

Proof. We already have a function $\mathrm{Div}_C(X) = \mathrm{IFI} \to \mathrm{Pic}$ (IFI: invertible frational ideals) given by $(\mathcal{L} \subseteq \mathcal{K}) \mapsto \mathcal{L}$. This map is an homomorphism. It is also onto: choosing a trivialization $\mathcal{L}|_U = \mathcal{O}|_U$ gives an isomorphism $\mathcal{L} \otimes_{\mathcal{O} \supseteq \mathcal{L}} \mathcal{K} \cong \mathcal{K}$. Now let's look at its kernel: it consits of sections of $\mathcal{K}^*/\mathcal{O}^*$ coming from $\mathcal{O} \subseteq \mathcal{K}$, which is just the same as the set of nonzero rational functions, which is $\operatorname{im}\Gamma(\mathcal{K}^*) = \Gamma(\mathcal{K}^*)/\Gamma(\mathcal{O}^*)$. ☐

In many scenarios, we can actually obtain explicit descriptions of the Picard group.

Theorem 1.1. *If* X *is locally factorial (i.e.* $\mathcal{O}_{X,x}$ *is always an UFD), then* $\mathrm{Div}_W(X) = \mathrm{Div}_C(X)$*.*

A remark about factoriality:

1. $k[x_1, \ldots, x_n]$ is an UFD, and a localization of an UFD is an UFD, from which it follows that $\mathbb{A}^n$ and $\mathbb{P}^n$ are locally factorial.
2. More generally, for a normal curve X, $U \subseteq X$, $\mathcal{O}(U)$ is a Dedekind domain (so that it is Noetherian, integrally closed, Krull dimension 1, equivalently, all frational ideals are invertible). In this case, $\mathcal{O}_{X,x}$ is a DVR, and therefore is an UFD.

Smoothness What we care in particular is that if X is smooth, then X is locally factorial. What is smoothness? One description is that if $x \in X$, then completion by the topology of the maximal ideal $\varprojlim_n \mathcal{O}_{X,x}/\mathfrak{m}_x^n = \widehat{\mathcal{O}_{X,x}}$ (the completed local ring) is isomorphic to $k[[x_1, \ldots, x_n]]$.

Proposition 2. *The following are true:*

1. $k[[x_1, \ldots, x_n]]$ *is a UFD.*
2. *If* A *is a Noetherian local ring such that its completion is an UFD, then* A *itself is an UFD.*

Remark 2. *The intuition that these local completion rings are the same as local charts for manifolds can be deceptive. For instance, the converse of b) may not be true, i.e.* A *is an UFD, but its completion is not. Also it may happen that* A *is an UFD, but* $A[[x]]$ *is not.*

Now observe that if X is a smooth variety, then $\mathcal{O}_{X,x}$ is a regular local ring, i.e. the maximal ideal $\mathfrak{m}_x$ is generated by a regular sequence, i.e. $x_1, \ldots, x_n$ such that x_i is not a zero divisor in the quotient $\mathcal{O}_{X,x}/(x_1, \ldots, x_{i-1})$ (in particular, x_1 is not a zero divisor). Observe that every Noetherian regular local ring is a UFD (AuslanderBuchsbaum theorem).

Proof of the Proposition. For the first statement, every finitely generated module has a finite resolution by free finitely generated modules, i.e. $0 \to F_n \to \ldots \to F_0 \to M \to 0$. For the second statement, this can be found as [Bou98, VII.7. Corollary 2]. If $I \subseteq A$ is Notherian local, then it is an intersection of principal ideals, and it has a finite free resolution, then it must be principal. □

Now back to the equivalence between Weil and Cartier divisors.

Proof of the Theorem. Consider the map $\mathrm{Div}_W(X) \to \mathrm{Div}_C(X)$ given by $[D] \mapsto J_D = \mathcal{O}(-D) \subseteq \mathcal{O} \subseteq \mathcal{K}$, where $\mathcal{O}(-D)$ denotes sheaf of functions vanishing on D. We need to know that J_D is locally principal. (The rest of this paragraph is slightly different from the original proof given in class.) Recall that when we have an UFD, every prime ideal of height one is principal. J_D is locally induced by a prime ideal of height 1 by definition, so when we pass to the stalk it is induced by (f_x) for some $f_x \in \mathcal{K}$. Now (f_x) and J_D only differ on components that do not pass x (as they agree on the stalk), which can only happen on finitely many other components, so after shrinking our local neighborhood we can have (f_x) agreeing with J_D on some neighborhood.

Now the map $[D] \mapsto J_D$ is clearly injective: enough to see that $[nD] \not\mapsto 0$ when $n \neq 0$, wlog when $n > 0$, but then the image is $J_D^n \subseteq J_D \neq 0$. It remains to check that the map is onto. First consider $\mathcal{L} \subseteq \mathcal{O}$, we want to find a Weil divisor D that goes to $\mathcal{L}$. Can asssume that we know this for all $\mathcal{L}'$ such that $\mathcal{L} \subsetneq \mathcal{L}' \subseteq \mathcal{O}$. Now pick $f \in \mathcal{L}$ such that locally $\mathcal{L} = (f)$, then we know that all components of Z_f have codimension 1, i.e. are Weil divisors. If D is such a component, then J_D contains $\mathcal{L}$; we can assume $J_D = (\varphi)$, then $\varphi^{-1}\mathcal{L}$ strictly contains $\mathcal{L}$ and is, by assumption, coming from some D', then $\mathcal{L}$ comes from $D + D'$. Finally, in the general case, $\mathcal{L} = (f)$ locally, where $f = \dfrac{\alpha}{\beta}$ where $\alpha, \beta \in \mathcal{O}(U)$, then we have shown that α comes from some D, β from some D', then f comes from $D - D'$. □

Example 1. *Suppose X is a normal curve, and $\mathcal{L} = (f)$, coming from $D = \sum_i n_i x_i$, where x_i are just points. So what are those values? The local multiplicity of x_i, i.e. n_i, is given by $\mathrm{val}_{x_i}(f)$.*

Another way to describe it is via $\mathcal{C} = \mathrm{coker}(\mathcal{O} \xrightarrow{f} \mathcal{O})$. Note that this is a coherent sheaf supported on the zeroes of f, so it splits as $\bigoplus_{x_i} \mathcal{C}_{x_i}$, and we claim that each has dim $\Gamma(\mathcal{C}_{x_i})$ finite, which equals the length of the sheaf.[1] To see this equivalence, consider the ideal sheaf $\mathcal{L} = J_x$, which comes from $-(x)$ by construction, then $\mathcal{L} = (f)$ is locally isomorphic to J_x^n (another way of saying the local ring is DVR), then it would come from $-(nx)$, but dim $\mathcal{O}_{X,x}/\mathfrak{m}_x^n = n$.

Remark 3. *In fact, for any irreducible X, we have a homomorphism in the other direction:* $\mathrm{Div}_C(X) \to \mathrm{Div}_W(X)$*. For instance, if X is a curve that is irreducible (but not necessarily normal), then we can send $\mathcal{L} = (f)$ to $\sum_i n_i x_i$, where $n_i = \dim\ \Gamma(\mathcal{C}_{x_i})$. If X is separated, irreducible, regular in codimension 1 (there exists $Z \subseteq X$, such that* codim $Z \geq 2$*, and $X - Z$ is regular), then this is an isomorphism.*

Let's do some easy examples.

Example 2. *The Picard group of $\mathbb{A}^n$ is trivial (every codimension 1 subvariety is given by a global function).*

Example 3. *What about $\mathbb{P}^n$? it is $\mathbb{Z}$, and is generated by $\{\mathcal{O}(d) \mid d \in \mathbb{Z}\}$.*

Proof. First see $\mathbb{Z}$ is contained in it because $\mathcal{O}(d_1) \otimes \mathcal{O}(d_2) = \mathcal{O}(d_1 + d_2)$, and that $\mathcal{O}(d) \neq \mathcal{O}$ when $d < 0$ because the global section vanishes for $d < 0$. The other inclusion holds because for any $D \subseteq \mathbb{P}^n$ of codimension 1, there is a homogeneous polynomial P of some degree d generating the homogeneous ideal vanishing on D, then $J_D = \mathcal{O}_{\mathbb{P}^n}(-d)$ by multiplication by P. □

[1] A coherent sheaf supported at x is an successive extension of $\mathcal{O}_x$, and the length of the sheaf is just the length of this filtration, i.e. number of extension steps needed.

Let's discuss the curve case in more detail. Let X be an irreducible, complete curve (not necessarily normal). Then one invariant of the divisor is the degree (which is $\deg(\sum_i n_i x_i) = \sum_i n_i$ for Weil divisor, and the degree of the corresponding image in Weil divisor if we have a Cartier divisor). Recall that Picard group is all Cartier divisors mod out all the principal divisors.

Proposition 3. *The degree of a principal divisor is zero.*

Thus we get a degree homomorphism from the Picard group to $\mathbb{Z}$.

References

[Bou98] N. Bourbaki. *Commutative Algebra: Chapters 1-7*. Vol. 1. Springer Science & Business Media, 1998.

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